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Simple connectedness of hyperplane complements in some geometries related to buildings

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ABSTRACT

For a class of parapolar spaces that includes the geometries $\mathsf{E}_{6,4}$, $\mathsf{E}_{7,7}$, and $\mathsf{E}_{8,1}$ with lines of size at least three, the metasymplectic spaces with lines of size at least four, and the polar line Grassmannians with lines of size at least four except $\mathsf{D}_{4,2}(3)$, we show that the subgraph of the point-collinearity graph induced on the complement of a hyperplane is simply connected. We also show that these parapolar spaces have Veldkamp lines.

1. Introduction

The present introduction contains definitions that we need in order to be able to state the results. In particular, Section 1.2 contains a description of the class of geometries dealt with in this paper.

The main results of the paper – Theorems 2.1 and 2.2 – are stated in Section 2, with the proof of Theorem 2.1 outlined in Section 2.2. At the end of Section 2 we describe the contents of the rest of the paper.

1.1. Parapolar spaces

A point-line space $\Gamma = (\mathcal{P}, \mathcal{L})$ is a set of points $\mathcal{P}$ together with a set $\mathcal{L}$ of subsets of $\mathcal{P}$; the elements of $\mathcal{L}$ are called lines and are required to have size at least two. Basic notions for point-line spaces not defined below can be found in A. Cohen's article in [2]. To simplify the notation we do not distinguish between a subspace (which is a set of points) of a point-line space $(\mathcal{P}, \mathcal{L})$ and the point-line space induced on it in $(\mathcal{P}, \mathcal{L})$. For a point $p \in \mathcal{P}$, the set of all points of Γ collinear with p, including p itself, will be denoted $p^\perp$. If $X \subseteq \mathcal{P}$, then we let $X^\perp = \bigcap_{p\in X} p^\perp$. A point-line space is

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said to have *thick* lines, if every line contains at least three points. A *geometric hyperplane* or just a *hyperplane* of Γ is a proper subspace of Γ that intersects every line of Γ at a nonempty set.

A point-line geometry is a bipartite graph with parts labeled "points" and "lines". Each point-line space gives rise to a point-line geometry which we also call a point-line space. In such a point-line geometry each line is incident with at least two points and no two lines are incident with exactly the same sets of points.

Before we can proceed, we need some definitions regarding graphs. Let $G = (V, E)$ be a graph. Here V is a set and E is a set of two-element subsets of V. For two vertices $p, q \in V$, a shortest walk from p to q in G is called a *geodesic* from p to q in G (see Section 1.3 for the definition of a walk). The *distance* from p to q in G is the length of a geodesic from p to q in G and will be denoted $\mathrm{d}_G(p,q)$. For a vertex p of G and an integer k, we denote $G_k(p)$ and $G_k^*(p)$ the sets of all vertices of G at distance k from p and at distance at most k from p. A set of vertices $X \subseteq V$ is *convex* in G if all vertices of every geodesic, that begins and ends in X, lie in X.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line space. A subspace of Γ is a *convex subspace* of Γ if it is convex in the point-collinearity graph of Γ. Let $X \subseteq \mathcal{P}$. The intersection of all convex subspaces of Γ containing X is called the *convex subspace closure* of X and is denoted $\langle\langle X\rangle\rangle$.

A *gamma space* is a point-line space in which, for every point-line pair $\{p, L\}$, $|p^\perp \cap L| \geqslant 2$ implies $L \subseteq p^\perp$.

A *parapolar space* Γ is a connected point-line space, which is a gamma space and has a family of convex subspaces $\mathcal{S}$, called *symplecta*, satisfying the following requirements: (1) each element of $\mathcal{S}$ is a nondegenerate polar space of rank at least 2; (2) every line of Γ lies in a symplecton; (3) every circuit of length 4 in the point-collinearity graph of Γ, that contains two points at distance 2, lies in a symplecton. If follows from the axioms (1) and (2) that every parapolar space is a partial linear space. In a parapolar space, symplecta of rank 2 are called *quads*. The least rank of a symplecton of Γ is called the *symplectic rank* of Γ. The largest rank of a maximal singular subspace of Γ is the *singular rank* of Γ.

Suppose Γ is a parapolar space, and let p and q be two points at distance two in the point-collinearity graph of Γ. Then there are two possibilities: either p and q have a unique common neighbor and are not contained in one symplecton, or p and q have at least two common neighbors and are contained in a symplecton. In the first case we say that the pair $\{p,q\}$ is *special*, and in the second case we say that it is *symplectic*. If $\{p,q\}$ is symplectic, then the unique symplecton containing $\{p,q\}$ will be denoted $\mathrm{S}(p,q)$. Similarly, if $A_1, \dots, A_k$ are points, lines, and planes (subspaces isomorphic to projective planes) of a parapolar space, all contained in a unique symplecton, that symplecton will be denoted $\mathrm{S}(A_1, \dots, A_k)$.

A parapolar space Γ is a *strong parapolar space* if every pair of points at distance 2 in the point-collinearity graph of Γ belongs to some symplecton, that is if Γ has no special pairs.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry, and assume that Γ is a point-line space. Suppose that all singular subspaces of Γ are projective spaces. The *residue of a point* p in Γ is the point-line geometry $\mathrm{Res}_\Gamma(p) = \Gamma_p = (\mathcal{L}_p, \Pi_p)$. The set of points of Γ_p is the set of all lines of Γ on p (denoted $\mathcal{L}_p$), the set of lines of Γ_p is the set of all singular subspaces of Γ on p that are projective planes (denoted Π_p), a point $L \in \mathcal{L}_p$ and a line $\pi \in \Pi_p$ are incident in Γ_p if and only if $L \subseteq \pi$ in Γ. If the point-collinearity graph of Γ is denoted Δ, then the point-collinearity graph of Γ_p will be denoted Δ_p. If all point residues of Γ are isomorphic to the same geometry Σ, then we say that Σ is the *local geometry* of Γ.

Suppose Γ is a parapolar space of symplectic rank at least three. Then by a theorem of Cooperstein [5] all singular subspaces of Γ are projective spaces. Each connected component of a point residue of Γ is a strong parapolar space (see Theorem 343(3) of Chapter 13 of [19]).

1.2. Hexagonic geometries

In this subsection we describe a class of parapolar spaces that will be the main subject of the present paper.

Table 1
Hexagonic geometries with thick lines characterized in [9].

Geometry	Local geometry	Known facts
metasymplectic space	dual polar space of rank 3	
$\mathsf{E}_{6,4}$ $\mathsf{E}_{7,7}$	$\mathsf{A}_{5,3}$ $\mathsf{D}_{6,6}$	For $\mathsf{A}_{n,k}$ and $\mathsf{D}_{n,n}$ it was shown that all hyperplanes arise from some absolutely universal embedding but not by the circuitry method (E.E. Shult [12,13])
$\mathsf{E}_{8,1}$	$\mathsf{E}_{7,1}$	The point-collinearity graph induced on the complement of every hyperplane in $\mathsf{E}_{7,1}$ is simply connected (E.E. Shult [16])
polar Grassmannian of lines of a nondegenerate polar space of possibly infinite rank $n \geqslant 4$	$L \times \mathbb{P}$	

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line space and let $\Delta = (\mathcal{P}, \mathcal{E})$ be the point-collinearity graph of Γ. In the present paper Γ will be called a *hexagonic geometry* if it satisfies the following axioms (the name "hexagonic" is due to B. Mühlherr).

(H0) Γ is a parapolar space of symplectic rank at least three.
(H1) For every point $p \in \mathcal{P}$ and every symplecton S of Γ, $p^{\perp} \cap S$ is either empty or contains a line (that is, $p^{\perp} \cap S$ cannot be a single point).
(H2) For every point residue $\Gamma_p = (\mathcal{L}_p, \Pi_p)$ and every point $x \in \mathcal{L}_p$, the set $\{q \in \mathcal{L}_p \mid \mathrm{d}_{\Delta_p}(x, q) \leqslant 2\}$ is a subspace of Γ_p that meets every line of $\mathcal{L}_p$ nontrivially.
(H3) For every point residue $\Gamma_p = (\mathcal{L}_p, \Pi_p)$ and every point $x \in \mathcal{L}_p$, there is a point $q \in \mathcal{L}_p$ such that $\mathrm{d}_{\Delta_p}(x, q) = 3$.

Let Γ be a point-line space with thick lines satisfying axioms (H0)–(H3). Suppose that Γ satisfies the additional requirement that either some symplecton has rank exactly three or the singular rank of Γ is finite. Then it was shown in [9] that Γ is one of the following geometries: $\mathsf{E}_{6,4}$, $\mathsf{E}_{7,7}$, $\mathsf{E}_{8,1}$, a metasymplectic space, or the polar Grassmannians of lines of a nondegenerate polar space of possibly infinite rank at least 4.

We list the geometries characterized in [9] in Table 1. The nodes of the diagrams are numbered as in [4]; the second number in the subscript shows which node of the diagram is selected as points, the lines are the flags of type consisting of all the nodes connected to the "points" node by a bond; L stands for a line and $\mathbb{P}$ for a nondegenerate polar space of rank at least 2. These geometries were first characterized by point-line axioms by Cohen and Cooperstein in [3] and [4], where they are the parapolar spaces of the conclusion of Theorem 2.

In the remainder of this subsection we state some properties of hexagonic geometries that will be used in the present paper.

Suppose Γ is a hexagonic geometry with point-collinearity graph Δ. By our definition of a parapolar space the geometry Γ is connected.

The axiom (H1) implies that, if $\{x, y\}$ is a symplectic pair, then no point adjacent to y can be at distance 3 from x.

The axioms (H0) and (H1) together imply that all point residues of Γ are connected and are strong parapolar spaces. The same axioms (H0) and (H1) also imply that Δ has point-diameter at most three (see Theorem 39 of [9]).

Lemma 1.1. *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a hexagonic geometry with point-collinearity graph Δ. Then the following hold.*

(H4) *For every point $p \in \mathcal{P}$, every geodesic of length* 2 *in Δ_p is extendable to a geodesic of length* 3.
(H5) *For every point $p \in \mathcal{P}$, the set $\Delta_2^*(p) = \{q \in \mathcal{P} \mid \mathrm{d}_\Delta(p, q) \leqslant 2\}$ is a hyperplane of Γ.*

Proof. The fact that (H0)–(H3) imply (H4) is stated as Theorem 21 in [9]. Unlike some other parts of Section 5 of [9], the proof of Theorem 21 does not require the singular subspaces of Γ to have finite projective rank.

We give a sketch of the proof of the fact that (H0)–(H3) imply (H5). The proof consists of three parts: (1) one can show, using Theorem 39 of [9], and methods similar to the method used in its proof that, for every point of p of Γ, the set $\Delta_2^*(p)$ meets every line of Γ; (2) using Lemma 3.4 (see Section 3.2), or just the axiom (H1) depending on the case, one can show that (H0)–(H3) imply that $\Delta_2^*(p)$ is a subspace of Γ; (3) using (H4) and Lemma 3.4 one can show that $\Delta_2^*(p)$ is a proper subset of $\mathcal{P}$. □

Remark. Recently, E.E. Shult described the geometries satisfying (H0), (H1), and the assumption that, if all symplecta have rank at least four, then all maximal singular subspaces have finite rank. These geometries are either hexagonic geometries listed in Table 1 or, else, strong parapolar spaces of point-diameter two [20].

1.3. Homotopy in graphs

Our treatment of homotopy here follows Shult [17], which was later incorporated into [19]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *walk* of length n from a to b in G is a sequence of vertices $w = (x_0, x_1, \dots, x_n)$, where $x_0 = a$ and $x_n = b$, such that $\{x_i, x_{i+1}\} \in E$ for all $i \in \{1, \dots, n-1\}$. We say that a and b are the initial and the terminal vertices of w. The *inverse walk* w^{-1} is the reverse sequence $w = (x_n, \dots, x_0)$. If $w_1 = (x_0, \dots, x_n)$ and $w_2 = (y_0, \dots, y_m)$ are two walks such that $x_n = y_0$, then the *concatenation* of w_1 and w_2 is the walk $w_1 \circ w_2 = (x_0, x_1, \dots, x_n, y_1, \dots, y_m)$. A *backtrack* is a walk of the form $w \circ w^{-1}$. A walk $w = (x_0, \dots, x_n)$ is a *circular walk* if $x_0 = x_n$; a circular walk has a beginning and an end. If $w = (x_0, \dots, x_n)$ is a circular walk then an *associate* of w is any walk of the form $(x_i, x_{i+1}, \dots, x_n, x_1, x_2, \dots, x_i)$.

Let G be a graph. Let $\mathcal{C}$ be a collection of circular walks of G and suppose that (1) $\mathcal{C}$ contains all walks of length 0 (that is, all walks of the form (v), $v \in V$); (2) for every $w \in \mathcal{C}$, the set $\mathcal{C}$ contains w^{-1} and all associates of w; (3) $\mathcal{C}$ contains all backtracks.

An *elementary $\mathcal{C}$-homotopy* of walks is a pair of walks (w_1, w_2) such that $w_1 = w_1' \circ p \circ w_1''$ and $w_2 = w_1' \circ q \circ w_1''$, for some walks w_1', w_1'', p, and q with $p \circ q^{-1} \in \mathcal{C}$. We say that a walk u is *$\mathcal{C}$-homotopic* to a walk v if, for some integer $n \geqslant 1$, there exists a sequence of elementary homotopies (w_0, w_1), $(w_1, w_2), \dots, (w_{n-1}, w_n)$ with $w_0 = u$ and $w_n = v$. The relation of being $\mathcal{C}$-homotopic is an equivalence relation on the set of all walks of G. The set all walks homotopic to a walk w is the *$\mathcal{C}$-homotopy class* of w. All walks in one homotopy class have common initial and terminal vertices.

A circular walk $w = (x_0, \dots, x_n)$, where $x_n = x_0$, is *$\mathcal{C}$-contractible* if it is $\mathcal{C}$-homotopic to the walk (x_0) of length 0. We say that a subgraph F of G is *$\mathcal{C}$-contractible* if F is connected, and every circular walk of F is $\mathcal{C}$-contractible in G. If the graph G itself is $\mathcal{C}$-contractible, then we say that G is *$\mathcal{C}$-simply connected*. When $\mathcal{C}$ is the set of all triangles, backtracks and walks of length 0, we say "homotopic", "contractible" and "simply connected" instead of $\mathcal{C}$-homotopic, $\mathcal{C}$-contractible, and $\mathcal{C}$-simply connected.

Suppose that G is a connected graph. A subgraph F of G *controls $\mathcal{C}$-homotopy* in G if every walk w in G, whose initial and terminal vertices are in F, is $\mathcal{C}$-homotopic to a walk in F.

Lemma 1.2. *(See Shult [17] and Lemma 18 of Shult [19].) Let G be a connected graph and let $\mathcal{C}$ be a collection of circular walks satisfying conditions* (1)–(3) *above.*

Suppose F is a subgraph of G. If F controls $\mathcal{C}$-homotopy in G, and F is $\mathcal{C}$-contractible, then G is $\mathcal{C}$-simply connected.

Proof. Suppose w is a circular walk in G with initial vertex x. Let y be any vertex of F, and let p be any walk from y to x in G. Then one can show that the circular walk $w' = p \circ w \circ p^{-1}$ is $\mathcal{C}$-contractible if and only if w is $\mathcal{C}$-contractible. If F controls $\mathcal{C}$-homotopy in G, then w' is $\mathcal{C}$-homotopic to a circular walk w'' of F (beginning and ending at y). If F is $\mathcal{C}$-contractible, then w'' is $\mathcal{C}$-contractible. □

1.4. Hyperplanes and embeddings

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line space. Suppose V is a vector space over a division ring k and let $\mathbb{P}(V)$ denote the corresponding projective space. A *projective embedding* of Γ over k or just an *embedding* $e : \Gamma \to \mathbb{P}(V)$ is an injective mapping from $\mathcal{P}$ to the set of points of the projective space $\mathbb{P}(V)$, such that every line of Γ is mapped bijectively onto a line of $\mathbb{P}(V)$, and the set of vectors of V comprising the image of $\mathcal{P}$ spans the entire vector space V. Point-line spaces that possess at least one projective embedding are called *embeddable*.

Let Γ be a point-line space, let V_1 and V_2 be vector spaces over a division ring k, and let $e_1 : \Gamma \to \mathbb{P}(V_1)$ and $e_2 : \Gamma \to \mathbb{P}(V_2)$ be two embeddings of Γ. Suppose that $t : V_1 \to V_2$ is a surjective k-semilinear map. Let $\mathrm{A}(V_1, \mathrm{Ker}(t))$ be the point-line space consisting of the points and lines of $\mathbb{P}(V_1)$ that intersect $\mathrm{Ker}(t)$ trivially, and let $\phi : \mathrm{A}(V_1, \mathrm{Ker}(t)) \to \mathbb{P}(V_2)$ be the morphism of point-line spaces induced by t. Suppose that $e_2 = e_1 \circ \phi$. Then we say that $\phi : e_1 \to e_2$ is a *morphism of embeddings over* k, and that the embedding e_2 is a *quotient* of e_1.

Suppose Γ is a point-line space and suppose that $e : \Gamma \to \mathbb{P}(V)$ is an embedding of Γ over a division ring k. Since the image points of every embedding are required to span the entire vector space, every morphism $\phi : e \to e$ must fix a basis of $\mathbb{P}(V)$ pointwise. Therefore ϕ comes from a bijective k-semilinear map and is invertible.

Suppose Γ is a point-line space, and suppose e_1 is an embedding of Γ over a division ring k. An embedding e of Γ over k is *relatively universal* with respect to e_1 if there is a morphism $\phi : e \to e_1$ such that, for every morphism $\phi' : e_2 \to e_1$, there exists a morphism $\phi'' : e \to e_2$ with $\phi = \phi'' \circ \phi'$. A theorem of Ronan (Proposition 3 of [11]) asserts that for every embedding e_1 there is a relatively universal embedding $\hat{e}_1$ of which e_1 is a quotient. The embedding $\hat{e}_1$ is called the universal hull of e_1.

Let Γ be a point-line space and let k be a division ring. An embedding u of Γ over k is *absolutely universal over* k if every embedding of Γ over k can be obtained as its quotient. That is, for every embedding e of Γ over k, there exists a morphism from u to e.

An absolutely universal embedding of a point-line space Γ, if it exists, is the universal hull of every embedding of Γ. In general, an embeddable geometry does not have to have an absolutely universal embedding. There is, however, an exception – every embeddable geometry Γ with constant line size 3 has an absolutely universal embedding: $\Gamma = (\mathcal{P}, \mathcal{L})$ embeds into the space V/V_0, where V is the GF(2) vector space with basis $\mathcal{P}$ and $V_0 = \langle \{a + b + c \mid \{a, b, c\} \in \mathcal{L}\} \rangle$.

Let Γ be a point-line space and let V be a vector space over a division ring. Suppose e is a projective embedding $\Gamma \to \mathbb{P}(V)$. Then every vector space hyperplane of $\mathbb{P}(V)$ gives rise to a geometric hyperplane of Γ: the set of points of Γ embedded in a vector space hyperplane is a geometric hyperplane of Γ. We say that a geometric hyperplane H of Γ *arises from the embedding* e if $e(H) = e(\mathcal{P}) \cap X$ for some vector space hyperplane X of $\mathbb{P}(V)$.

The following theorem of M.A. Ronan gives a sufficient condition for a geometric hyperplane to arise from every relatively universal embedding of Γ.

Theorem 1.3. *(See Ronan, Corollary 5 of [11].) Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be an embeddable point-line geometry with point-collinearity graph Δ and let H be a hyperplane of Γ.*

Suppose that every triangle of the point-collinearity graph of Γ lies in a projective plane which is a full subgeometry of Γ (the lines of the plane are full lines of Γ). Suppose also that $\Delta|(\mathcal{P} - H)$ is simply connected.

Then H arises from every relatively universal embedding of Γ.

If Γ is an embeddable point-line space with constant line size three, then by Corollary 2 of Ronan [11] every geometric hyperplane of Γ arises from the absolutely universal embedding of Γ.

1.5. Veldkamp points and Veldkamp lines

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line space. We say that Γ *has Veldkamp points* if the following condition holds.

(1) All geometric hyperplanes of Γ are maximal subspaces of Γ.

Suppose now that Γ is a point-line space and Γ has Veldkamp points. Then we say that Γ *has Veldkamp lines* if Γ satisfies the following condition.

(2) If H_1, H_2 and H_3 are hyperplanes of Γ such that $H_1 \cap H_2 \subseteq H_3$ but $H_1 \nsubseteq H_3$ then $H_1 \cap H_3 \subseteq H_2$.

Suppose Γ is a point-line space and let $\mathcal{V}$ be the set of all hyperplanes of Γ. If Γ has Veldkamp lines, then $\mathcal{V}$ can be given the structure of a linear space, called the *Veldkamp space* of Γ.

The points of the Veldkamp space are the hyperplanes of Γ, the lines of the Veldkamp space are the intersections of pairs of distinct hyperplanes; a point H and a line $H_1 \cap H_2$ are incident if and only if $H_1 \cap H_2 \subseteq H$.

We have the following criterion for existence of Veldkamp lines (see Lemma 4.1 of [15]).

Suppose Γ is a point-line space with point-collinearity graph Δ. If Γ satisfies the following two conditions, then Γ has Veldkamp lines.

(V1) For every hyperplane H of Γ, the graph $\Delta|(\mathcal{P} - H)$ is connected.
(V2) For every pair of hyperplanes H_1, H_2 of Γ the graph $\Delta|(H_1 - (H_1 \cap H_2))$ is connected.

Condition (V1) is equivalent to existence of Veldkamp points.

Let Γ be a point line space and suppose that Γ has Veldkamp lines. Suppose $e : \Gamma \to \mathbb{P}(V)$ is a projective embedding of Γ. Then every geometric hyperplane of Γ spans a subspace of $\mathbb{P}(V)$ of codimension at most 1, and every intersection of two hyperplanes spans a subspace of $\mathbb{P}(V)$ of codimension at most 2. By a theorem of Shult (Theorem 3 of [14]), the set of all hyperplanes of Γ arising from e is a subspace of the Veldkamp space $\mathcal{V}$, and the geometry induced on it in $\mathcal{V}$ is a projective space.

One can use Theorem 7 of Shult [14] (that requires existence of Veldkamp lines, and existence of a finite-dimensional embedding such that all geometric hyperplanes arise from it), together with Proposition 3 of [11] and Theorem 1.3, to show existence of an absolutely universal embedding for an embeddable geometry (see [14,6,7,16]). For an entirely different approach to showing existence of absolutely universal embeddings, based on Proposition 3 of Ronan [11], see [8].

2. Summary of the results

In Section 2.1 we state the main results of the paper. In Section 2.2 we outline the proof of Theorem 2.1.

We use the following notation. Let $G = (V, E)$ be a graph. For $V' \subseteq V$, we denote $G|V'$ the subgraph of G induced on V'. Suppose $G' = (V', E')$ is a subgraph of G and suppose $V_1 \subseteq V$. We denote $G' - V_1$ the graph $G'|(V' - (V' \cap V_1))$.

2.1. Statement of the results

Let Γ be a hexagonic geometry. We consider the following condition.

Condition (A).

(i) All symplecta of Γ have rank at least 4 and all lines are thick or
(ii) some symplecta of Γ have rank 3, every line contains at least 4 points, and Γ is not the geometry $D_{4,2}(3)$.

The geometries listed in Table 1 that do *not* satisfy condition (A) are as follows: the metasymplectic spaces with lines of size three, the polar line Grassmannians with lines of size three, and the polar line Grassmannian $D_{4,2}(3)$ (whose lines have size four).

The reasons for considering condition (A) are explained in remark (1) after Lemma 2.6.

We can now state our main results.

Theorem 2.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a hexagonic geometry and let Δ denote the point-collinearity graph of Γ. Suppose that Γ satisfies condition* (A).

Let H be a hyperplane of Γ. Then the graph $\Delta|(\mathcal{P} - H)$ is simply connected.

Theorem 2.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a hexagonic geometry. Suppose that Γ has thick lines. Then Γ has Veldkamp lines.*

Let Γ be a parapolar space. Then every triangle of the point-collinearity graph of Γ lies in a singular subspace of Γ. If Γ has symplectic rank at least three, then every singular subspace of Γ is a projective space. Therefore, combining Theorem 2.1 with Theorem 1.3, we immediately obtain the following.

Corollary 2.3. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a hexagonic geometry. Suppose that Γ satisfies condition* (A).

If Γ is embeddable, then all geometric hyperplanes of Γ arise from every relatively universal embedding of Γ.

Let Γ be an embeddable hexagonic geometry of finite singular rank. Then it was shown in [8] that Γ has an absolutely universal embedding. Therefore, if Γ satisfies condition (A), then by Corollary 2.3 all hyperplanes of Γ arise from its absolutely universal embedding.

2.2. Proof of Theorem 2.1

Theorem 2.1 will be an immediate consequence of Lemma 1.2 and of the following two propositions.

Proposition 2.4. *Suppose that the hypothesis of Theorem* 2.1 *holds.*

Let $p \in \mathcal{P} - H$. Then the graph $\Delta|(\Delta_2^*(p) - H)$ is contractible in $\Delta - H$.

Proposition 2.5. *Suppose that the hypothesis of Theorem* 2.1 *holds.*

Then the following statements are true.

(i) *The graph $\Delta - H$ is connected.*
(ii) *For every point p in $\mathcal{P} - H$, the subgraph of $\Delta|(\Delta_2^*(p) - H)$ controls homotopy in $\Delta - H$.*

The proof of Proposition 2.5 uses Proposition 2.4. The proof of Proposition 2.4 relies on Lemma 2.6. To state Lemma 2.6 we need the notion of deep point introduced in [15].

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line space. Suppose X is a proper subset of $\mathcal{P}$. We say that a point $x \in X$ is a *deep point* of X if $x^{\perp} \subseteq X$.

Lemma 2.6. *Assume that the hypothesis of Theorem* 2.1 *holds.*

Suppose that

(i) *$h \in \mathcal{P} - H$ or*
(ii) *$h \in H$ and h is not deep in H.*

Then the subgraph $\Delta|(h^{\perp} - H)$ of $\Delta - H$ is contractible in $\Delta - H$.

Remarks. (1) The proof of Lemma 2.6 uses the full strength of condition (A). More precisely, the proof is divided into two cases, corresponding to (i) and (ii) of condition (A). The proof of case corresponding to condition (A)(ii) relies on Lemma 7.12, which states that, if Q is a generalized quadrangle with at least four points on every line, and Q is not a 4×4 grid, then the point-collinearity graph of Q minus the union of any two hyperplanes is connected.

(2) When the local geometry of Γ is $E_{7,1}$, its point-collinearity graph minus a hyperplane, and therefore the graph $\Delta|(h^{\perp} - H)$, are known to be contractible (E.E. Shult [16]).

(3) By Lemma 2.6 the subgraph $\Delta|(h^{\perp} - H)$ is contractible in $\Delta - H$, but as a graph it might not be simply connected. For example, when Γ is a metasymplectic space, the local geometry $\mathrm{Res}_{\Gamma}(h)$ is a dual polar space of rank 3 and contains no planes. The point-collinearity graph of $\mathrm{Res}_{\Gamma}(h)$ and the subgraph $\Delta|(h^{\perp} - H)$ are not simply connected in this case. However, if $\mathcal{C}$ is a set of circular walks containing all circuits of length 4, then $\Delta|(h^{\perp} - H)$ is $\mathcal{C}$-simply connected. Since every circuit of length 4 of Δ lies in a symplecton of Γ, it is contractible in $\Delta - H$ by Theorem 3.1. Therefore, $\Delta|(h^{\perp} - H)$ is contractible in $\Delta - H$.

The contents of the rest of the paper are as follows.

Section 3 contains auxiliary results and definitions. In Section 4 we prove existence of opposite lines and existence of Veldkamp points for hexagonic geometries with thick lines.

In Sections 5 and 6 we prove Propositions 2.4 and 2.5 respectively, and in Section 7 we prove Lemma 2.6. Combined, these results constitute a proof of Theorem 2.1.

In Section 8 we prove Theorem 2.2.

3. Auxiliary results

In this section we collect some necessary results and additional definitions for later use. More specifically, Section 3.1 contains some known results regarding parapolar spaces, Section 3.2 describes an important property of hexagonic geometries, and Section 3.3 contains additional definitions.

3.1. Some known results for polar and parapolar spaces

In this subsection we state three results of E.E. Shult and A. Pasini on hyperplanes of polar spaces and strong parapolar spaces that will be used frequently.

Theorem 3.1. *(See Shult [18] or Theorem 135 of [19], see also Lemma 2.5.1 of [7]; Pasini [10].) Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a polar space with point-collinearity graph Δ. Suppose that Γ has rank at least 3, and suppose that Γ has thick lines.*

Let H be a hyperplane of Γ. Then the graph $\Delta|(\mathcal{P} - H)$ is simply connected.

Let Γ be a polar space and suppose that Γ is nondegenerate. Then, for every $p \in \mathcal{P}$, the set $p^{\perp}$ is a hyperplane of Γ and, for all $p, q \in \mathcal{P}$ with $p \neq q$, we have $p^{\perp} \neq q^{\perp}$.

Suppose now that Γ is nondegenerate polar space with thick lines. Then by Theorem 3.1 (see also Corollary 3.3 below) every proper subspace of Γ has at most one deep point.

For other facts regarding polar spaces see [1] and [21].

According to our definition of parapolar space given in Section 1.1, a parapolar space is allowed to have symplectic rank 2. Therefore our strong parapolar space corresponds to the polarized space of [15] and we have the following.

Theorem 3.2. *(See Shult [15, Corollary 6.7].) A strong parapolar space with thick lines is never the union of two proper subspaces.*

From Theorem 3.2 we immediately obtain the next corollary.

Corollary 3.3. *(See Shult [15].) Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a strong parapolar space with point-collinearity graph Δ. Suppose that Γ has thick lines.*

Let H be a hyperplane of Γ. Then the graph $\Delta|(\mathcal{P} - H)$ is connected. That is, Γ has Veldkamp points.

3.2. A property of hexagonic geometries

In this subsection we prove a result regarding the structure of hexagonic geometries and, then, state its corollary that will be used in Sections 4, 6, and 8.

Lemma 3.4. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a hexagonic geometry, and let $p \in \mathcal{P}$. Suppose $\{x, p\}$ and $\{y, p\}$ are two distinct special pairs in Γ, and let a and b be two points such that $\{a\} = x^\perp \cap p^\perp$ and $\{b\} = y^\perp \cap p^\perp$. If x is collinear with y then either $a = b$ or $a \neq b$ and a is collinear with b.*

Proof. Assume $a \neq b$. Suppose S is a symplecton on $\{x, y\}$, such that $p^\perp \cap S \neq \emptyset$. Then by the axiom (H1) the set $p^\perp \cap S$ contains a line L, and $x^\perp \cap L$ and $y^\perp \cap L$ are nonempty. Since $\{p, x\}$ and $\{p, y\}$ are special, we have $x^\perp \cap L = \{a\}$, $y^\perp \cap L = \{b\}$ and, therefore, a is collinear with b. Thus we only need to show that such a symplecton S exists.

If $\{a, y\}$ is symplectic we can put $S = \mathrm{S}(a, y)$, if $\{b, x\}$ is symplectic we can put $S = \mathrm{S}(b, x)$. Suppose both $\{a, y\}$ and $\{b, x\}$ are special. Pick any plane π on $\langle x, y\rangle$. By (H2) there is a line $L \subseteq \pi$ containing x such that $\langle\langle a, L\rangle\rangle$ is a symplecton; similarly there exists a line $M \subseteq \pi$ on y such that $\langle\langle b, M\rangle\rangle$ is a symplecton.

Let $S_1 = \langle\langle a, L\rangle\rangle$ and let $S_2 = \langle\langle b, M\rangle\rangle$. Let L' be a line in $p^\perp \cap S_1$ on a, and let M' be a line in $p^\perp \cap S_2$ on b; such lines exist by (H1). If $L' \cap \pi$ were nonempty the point of their intersection would have to coincide with both a and b, in particular a would be equal to b, which is not the case. Therefore assume L' and M' do not intersect π.

Consider the pair L, L' inside S_1. We have L' is not contained in $x^\perp$, since $\{x, p\}$ is special, and L is not contained in $a^\perp$, since $\{a, y\}$ is special. Hence L and L' are opposite lines inside S_1 (that is $L^\perp \cap L' = \emptyset$). Similarly, M and M' are opposite inside S_2.

Let $\{c\} = L \cap M$ and $\{a'\} = c^\perp \cap L'$, $\{b'\} = c^\perp \cap M'$. If $a' = b'$, let T be the symplecton $\mathrm{S}(a, p, b)$. If $a' \neq b'$ let $T = \mathrm{S}(c, p)$. In both cases T is a symplecton containing p and such that $x^\perp \cap T$ and $y^\perp \cap T$ are both nonempty.

Let L_x be a line in $x^\perp \cap T$ and let L_y be a line in $y^\perp \cap T$. If $L_x \cap L_y \neq \emptyset$ put $S = \mathrm{S}(L_x, y, x)$ (the convex subspace closure of $L_x \cup \{y\}$ is a symplecton since $L_x \nsubseteq y^\perp$ or we would have $\{a\} = p^\perp \cap L_x = \{b\}$ against the current hypothesis).

If $L_x \cap L_y = \emptyset$ (this can happen only if $a' = b'$), then there exists a line N in T with $N \cap L_x$ and $N \cap L_y$ both nonempty. Then $N \nsubseteq x^\perp \cap y^\perp$, since otherwise a and b would both be equal to $p^\perp \cap N$, therefore we can let $S = \mathrm{S}(N, x, y)$.

In both cases S is a symplecton containing $\langle x, y\rangle$, and $p^\perp \cap S$ is nonempty because S contains a line, L_x or N, of the symplecton T, and $p \in T$. □

Corollary 3.5. *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a hexagonic geometry and let Δ be the point-collinearity graph of Γ.*

Let p and q be two points of Γ at distance 3 from each other, and let $U = p^\perp \cap \Delta_2^*(q)$ and $V = q^\perp \cap \Delta_2^*(p)$.

Let $\Gamma|U$ and $\Gamma|V$ denote the point-line spaces induced in Γ on U and V. Then the following statements hold.

(i) *$\Gamma|U$ and $\Gamma|V$ are subspaces of Γ isomorphic to $\mathrm{Res}_\Gamma(p)$ and $\mathrm{Res}_\Gamma(q)$.*
(ii) *The map $\phi : U \to V$, that takes each $u \in U$ to the unique point $v \in V$ such that $\{v\} = u^\perp \cap q^\perp$, determines an isomorphism $\Gamma|U \to \Gamma|V$.*

Proof. By Lemma 1.1 condition (H5) holds, therefore $\Delta_2^*(p)$ and $\Delta_2^*(q)$ are hyperplanes of Γ. This implies (i).

We now prove (ii). Suppose u_1 and u_2 are distinct points of U, and let $v_1 = \phi(u_1)$ and $v_2 = \phi(u_2)$. If $v_1 = v_2$ then the pair $\{v_1, p\}$ is symplectic (since $u_1 \neq u_2$), which is impossible by the axiom (H1). Therefore,

(1) ϕ maps U to V injectively.

Suppose $v \in V$. Let $u \in U$ be such that $v^{\perp} \cap U = \{u\}$. Then $\phi(u) = v$. Therefore

(2) ϕ maps U to V surjectively.

We now show that

(3) ϕ maps the lines of $\Gamma|U$ to lines of $\Gamma|V$.

Suppose that L is a line of U. Let u_1 and u_2 be distinct points of L. Let $v_1 = \phi(u_1)$ and $v_2 = \phi(u_2)$. By Lemma 3.4 $v_1 \in v_2^{\perp}$. Since ϕ is injective on points, v_1 and v_2 are distinct collinear points of V.

Let $N = \langle v_1, v_2 \rangle$ and let $S = \mathrm{S}(u_1, u_2, v_1, v_2)$. Then L and N are opposite lines of S. Therefore, ϕ maps L bijectively onto N. This proves (3).

Combining (3) with (1) we obtain that

(4) ϕ maps the set of lines of $\Gamma|U$ injectively to the set of lines of $\Gamma|V$.

It remains to show that

(5) every line of $\Gamma|V$ is the image of a line of $\Gamma|U$ under ϕ.

To prove (5), we interchange the roles of U and V in the proof of (3). That is, let $L = \langle u_1, u_2 \rangle$ be a line of $\Gamma|V$, let $\{v_1\} = u_1^{\perp} \cap U$, and let $\{v_2\} = u_2^{\perp} \cap U$. Then the line $N = \langle v_1, v_2 \rangle$ is mapped by ϕ to L. $\square$

3.3. Additional definitions

This subsection consists of four independent parts labeled 1–4.

1. We define hexagons of the first and second kind. In general, these circuits have length more than six.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a parapolar space and let Δ be the point-collinearity graph of Γ. Suppose H is a hyperplane of Γ.

Suppose that $C = \mathrm{w}(p, a) \circ (a, x, a') \circ \mathrm{w}(a', p)$ is a circular walk in $\Delta - H$, such that $a, a' \in \Delta_2^*(p)$, $x \in \mathcal{P} - \Delta_2^*(p)$, and $\mathrm{w}(p, a)$ and $\mathrm{w}(a', p)$ are walks from p to a and from a' to p in $\Delta_2^*(p) - H$. Then we say that C is a *hexagon of the first kind* in $\Delta - H$.

A circular walk in $\Delta - H$ will be called a *hexagon of the second kind* if it has the form $\mathrm{w}(x, u) \circ (u, v) \circ \mathrm{w}(v, y) \circ (y, x)$, where $\mathrm{d}_\Delta(x, u) = 2 = \mathrm{d}_\Delta(y, v)$, $\mathrm{d}_\Delta(x, v) = 3$, $\mathrm{w}(x, u)$ is a walk from x to u in $\Delta_2^*(x) - H$, and $\mathrm{w}(v, y)$ is a walk from v to y in $\Delta_2^*(v) - H$.

2. We shall need the notion of opposite lines.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a parapolar space and let Δ denote the point-collinearity graph of Γ. Suppose Δ has finite diameter n and suppose that, for every $p \in \mathcal{P}$, the set $\Delta_{n-1}(p)$ is a hyperplane of Γ.

Let L and M be lines of Γ. One can check that there are three possibilities:

(1) $M \subseteq \Delta_{n-1}^*(p)$ for every $p \in L$ and $L \subseteq \Delta_2^*(q)$ for every $q \in M$.
(2) There are unique points $p \in L$ and $q \in M$, such that $M \subseteq \Delta_{n-1}^*(p)$ and $L \subseteq \Delta_{n-1}^*(q)$.
(3) For every point $p \in L$, the set $\Delta_{n-1}^*(p) \cap M$ is a single point and, for every point $q \in L$, the set $\Delta_{n-1}^*(q) \cap L$ is a single point.

If two lines L and M of Γ satisfy (3), then we say that L and M are *opposite* lines of Γ. Being opposite is a symmetric relation on lines.

Suppose that Γ is a hexagonic geometry. Then by Lemma 1.1 Γ satisfies (H5), therefore the definition of opposite lines applies with $n = 3$.

3. We assign names to several conditions that will be used repeatedly.

Suppose that $\Gamma = (\mathcal{P}, \mathcal{L})$ is a parapolar space and let Δ be the point-collinearity graph of Γ.

(C0) Every plane of Γ lies in a symplecton.
(C1) All point residues of Γ are connected.

For condition (C2) suppose that Γ satisfies condition (H5).

(C2) For every $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$ such that $L \nsubseteq \Delta_2^*(x)$, there exists a line M on x opposite to L.

For conditions (C3) and (C4) assume that H is a hyperplane of Γ.

(C3) For every point $p \in \mathcal{P} - H$, the subgraph of $\Delta|(\Delta_2^*(p) - H)$ is contractible in $\Delta - H$.
(C4) All hexagons of the second kind in $\Delta - H$ are contractible.

Condition (C1) implies condition (C0). To prove this, suppose that $\Gamma = (\mathcal{P}, \mathcal{L})$ is a parapolar space satisfying (C1). Let π be a plane of Γ, let L be a line of π, and let S be symplecton of Γ on L. If $\pi \subseteq S$, then we are done.

Suppose $\pi \nsubseteq S$ and suppose that S has rank at least three. Since S is a convex subspace of Γ, $\pi^\perp \cap S$ is a singular subspace of S. Since S is a nondegenerate polar space, $L^\perp \cap S$ is not a singular subspace of S. Therefore there exists a plane π' of Γ on L such that $\pi' \nsubseteq \pi^\perp$. Then $\langle\langle \pi, \pi' \rangle\rangle$ is a symplecton of Γ containing π.

Suppose $\pi \nsubseteq S$ and suppose that S has rank two. Let p be a point of L. Since S is a nondegenerate polar space, $p^\perp \cap S$ is not a singular subspace of S. Therefore there exists a line M of S on p such that $M \nsubseteq L^\perp$.

By (C1) $\mathrm{Res}_\Gamma(p)$ is connected. Let w be a shortest walk from L to M in $\mathrm{Res}_\Gamma(p)$. Since $M \nsubseteq L^\perp$, the walk w has length at least two. Let N be the vertex of w at distance two in w from L. Then $S' = \langle\langle L, N \rangle\rangle$ is a symplecton of rank at least three, and S' shares the line L with π. Therefore we are back to one of the two previous cases. This shows that (C1) implies (C0).

4. We define certain walks in the complement of a hyperplane.

Let Γ be a parapolar space and let Δ denote the point-collinearity graph of Γ. Assume that Γ has symplectic rank at least three, has thick lines, and satisfies (C1).

Suppose H is a hyperplane of Γ and let x and y be two points in $\mathcal{P} - H$ with $\mathrm{d}_\Delta(x, y) \leqslant 2$. We define $\mathrm{P}(x, y)$ to be the following walk from x to y in $\Delta - H$.

(1) If $x = y$, then $\mathrm{P}(x, y) = (x)$.
(2) If x is collinear with y but $x \neq y$, then $\mathrm{P}(x, y) = (x, y)$.
(3) If $\mathrm{d}_\Delta(x, y) = 2$ and the pair $\{x, y\}$ is symplectic, then $\mathrm{P}(x, y)$ is a walk from x to y in the graph $\Delta|(\mathrm{S}(x, y) - H)$.
(4) If $\mathrm{d}_\Delta(x, y) = 2$ and the pair $\{x, y\}$ is special, then $\mathrm{P}(x, y)$ is a walk from x to y in the graph $\Delta|(a^\perp - H)$, where $\{a\} = x^\perp \cap y^\perp$.

Let $x, y \in \mathcal{P} - H$ and suppose that $\mathrm{d}_\Delta(x, y) \leqslant 2$. We claim that there exists a walk $\mathrm{P}(x, y)$. Furthermore, if Γ is a hexagonic geometry satisfying condition (A), then all walks that can be chosen as $\mathrm{P}(x, y)$ lie in one homotopy class.

In cases (1) and (2) it is clear that the walk $\mathrm{P}(x, y)$ exists and is unique.

In case (3), since all lines of Γ are thick and Γ has symplectic rank at least three, the graph $\Delta|(\mathrm{S}(x, y) - H)$ is simply connected by Theorem 3.1. Therefore, there is a walk from x to y in $\mathrm{S}(x, y) - H$, and all walks from x to y inside $\mathrm{S}(x, y) - H$ lie in one homotopy class.

In case (4), if $a \notin H$, then it is clear that the graph $\Delta|(a^\perp - H)$ is connected and simply connected. Suppose $a \in H$. Since by hypothesis condition (C1) and therefore also condition (C0) hold, $\mathrm{Res}_\Gamma(a)$ is a strong parapolar space. Since all lines of Γ are thick, the lines of $\mathrm{Res}_\Gamma(a)$ are thick. Therefore, by Corollary 3.3 $\mathrm{Res}_\Gamma(a)$, minus its hyperplane determined by H, is connected. This shows that the walk $\mathrm{P}(x, y)$ exists.

Suppose that Γ is a hexagonic geometry satisfying condition (A). Then by Lemma 2.6 all walks from x to y in the subgraph $\Delta|(a^{\perp} - H)$ of $\Delta - H$ are homotopic to each other in $\Delta - H$. However, this might not be true in a more general parapolar space.

We observe that case (4) does not occur if Γ is a strong parapolar space.

4. Opposite lines and existence of Veldkamp points in hexagonic geometries

In this section we use opposite lines to show existence of Veldkamp points for all hexagonic geometry with thick lines (Proposition 4.2).

Before stating the results, we make the following observation. Suppose that Γ is a hexagonic geometry. Let L and M be opposite lines of Γ. Let $p \in L$ and let $\{q\} = \Delta_2^*(p) \cap M$. Then the axiom (H1) implies that the pair $\{p, q\}$ is special.

Lemma 4.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a hexagonic geometry and let Δ denote the point-collinearity graph of Γ.*

Suppose that $x, y \in \mathcal{P}$ are such that $\mathrm{d}_\Delta(x, y) = 3$. If L is a line of Γ on x, then there exists a line M of Γ on y opposite to L.

Proof. Let $X = x^{\perp} \cap \Delta_2^*(y)$ and let $Y = y^{\perp} \cap \Delta_2^*(x)$. By Corollary 3.5(i) the sets X and Y are subspaces of Γ isomorphic to $\mathrm{Res}_\Gamma(x)$ and $\mathrm{Res}_\Gamma(y)$ respectively. Let $\phi : \Gamma|X \to \Gamma|Y$ be the isomorphism of point-line space $\Gamma|X$ and $\Gamma|Y$ as in Corollary 3.5(ii).

Let l be the unique point of L such that $\{l\} = L \cap \Delta_2^*(y)$. By (H4) applied to $\mathrm{Res}_\Gamma(y)$ the subspace Y contains a point m, such that the distance from $\phi(l)$ to m in the point-collinearity graph of Y is 3. Let $M = \langle y, m\rangle$.

We claim that the lines L and M are opposite. It suffices to show that $M \nsubseteq \Delta_2^*(l)$ and $L \nsubseteq \Delta_2^*(y)$, the latter being obvious since $\mathrm{d}_\Delta(x, y) = 3$. Suppose by way of contradiction that $\mathrm{d}_\Delta(l, m) \leqslant 2$. We have $\mathrm{d}_\Delta(l, m) \neq 1$, since $m \neq \phi(l)$ by our choice of m. Therefore $\mathrm{d}_\Delta(l, m) = 2$.

Suppose $\{l, m\}$ is a symplectic pair. Then by (H1) the intersection $y^{\perp} \cap \mathrm{S}(l, m)$ contains a line N on m and $l^{\perp} \cap N \neq \emptyset$. Since $\phi(l)$ is the unique point of $l^{\perp} \cap y^{\perp}$, this implies $\phi(l) \in N$. In particular, $\phi(l)$ is collinear with m, against the choice of m.

Suppose that $\{l, m\}$ is a special pair and let $z \in \mathcal{P}$ be such that $\{z\} = l^{\perp} \cap m^{\perp}$. By Lemma 3.4 $\phi(l)$ is collinear with z. But this implies that the pair $\{\phi(l), m\}$ is symplectic, a contradiction with the choice of m. This proves that M is opposite to L. $\square$

The next Proposition 4.2 will be used in the proof of Lemma 6.5, that will be applied to both hexagonic geometries and local geometries of hexagonic geometries.

Proposition 4.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a parapolar space and let Δ be the point-collinearity graph of Γ.*

Suppose that Γ has thick lines and satisfies conditions (H5), (C1), *and* (C2).

Let H be a hyperplane of Γ. Then the graph $\Delta - H$ is connected.

Proof. Let x and y be two points in $\mathcal{P} - H$. We need to show that there is a walk from x to y in $\Delta - H$. This is clearly true if $x \in y^{\perp}$.

Suppose $\mathrm{d}_\Delta(x, y) = 2$. Let a be a point in $x^{\perp} \cap y^{\perp}$. If $a \notin H$, then (x, a, y) is the required walk. Suppose $a \in H$. By condition (C1) the point residue $\mathrm{Res}_\Gamma(a)$ is a strong parapolar space. By hypothesis Γ has thick lines, therefore by Corollary 3.3 the space $\mathrm{Res}_\Gamma(a)$ minus its hyperplane, consisting of all lines of Γ on a that lie in H, is connected. This implies that there exists a walk from x to y in $a^{\perp} - H$.

Suppose that $\mathrm{d}_\Delta(x, y) = 3$. Then by (C2) there exist opposite lines L containing x and M containing y. Since all lines are thick, there is a pair of points $a \in L - H$ and $b \in M - H$ such that $\mathrm{d}_\Delta(a, b) = 2$. By the previous case a and b lie in one connected component of $\Delta - H$, therefore so do x and y. $\square$

5. Proof of Proposition 2.4

The structure of the present section is as follows.

Corollary 5.1 of Section 5.1 and Lemma 5.2 of Section 5.2 are the main ingredients of the proof of Proposition 2.4. The proofs of both use Lemma 2.6; the proof of Lemma 5.2 uses also Corollary 5.1.

The proof of Proposition 2.4 is given in Section 5.3.

5.1. Contractibility of subgraphs $\mathcal{S}(x) - H$ of $\Delta - H$

We need the following definition. For a point $x \in \mathcal{P}$, let $\mathrm{S}(x)$ denote the set of all points of Γ that lie in symplecta on x. We denote by $\mathcal{S}(x)$ the subgraph of Δ with vertex set $\mathrm{S}(x)$ and the set of edges consisting of all edges of Δ lying in symplecta on x. That is, p and q are adjacent in $\mathcal{S}(x)$ if and only if they are collinear points of some symplecton S containing x.

In general $\mathcal{S}(x)$ is not an induced subgraph of Δ (although in the case of geometries of type F_4 it is). This can be seen as follows.

Let S_1 and S_2 be two distinct symplecta on x, and let $v \in S_1$ and $w \in S_2$ be two points with $\mathrm{d}_\Delta(x, v) = 2$ and $\mathrm{d}_\Delta(x, w) = 2$. Every symplecton of Γ is uniquely determined by any pair of points at distance 2 in it. In particular, $S_1 = \mathrm{S}(x, v)$, $S_2 = \mathrm{S}(x, w)$, and no symplecton on x contains both v and w. This implies that v and w are *not* adjacent in $\mathcal{S}(x)$ even though they may be collinear in Γ.

The subgraph induced in $\mathcal{S}(x)$ on the set $x^\perp$ is $\Delta|x^\perp$: the geometry Γ is a parapolar space of symplectic rank at least three, therefore every plane of Γ on x, and hence every edge of Δ inside $x^\perp$, lies in a symplecton on x.

We have the following corollary of Lemma 2.6.

Corollary 5.1. *Suppose that the hypothesis of Theorem* 2.1 *holds.*
Then the following statements are true.

(i) *For every $p \in \mathcal{P} - H$, the graph $\mathcal{S}(p) - H$ is simply connected.*
(ii) *For every nondeep point $h \in H$, every connected component of the graph $\mathcal{S}(h) - H$ is contractible in $\Delta - H$.*

Remark. Conclusion (i) of Corollary 5.1 still holds if condition (A) is replaced with the weaker hypothesis that all lines of Γ are thick.

Proof of Corollary 5.1. (i) Suppose $p \notin H$ and let C be a circular walk in $\mathcal{S}(p) - H$. From the discussion immediately preceding this corollary we see that C has the form $\mathrm{w}(a_0, a_1) \circ \mathrm{w}(a_1, a_2) \circ \cdots \circ \mathrm{w}(a_{n-1}, a_n)$ where $a_n = a_0$ and, for every $i \in \{0, \ldots, n-1\}$, $a_i \in p^\perp$ and $\mathrm{w}(a_i, a_{i+1})$ is a walk from a_i to a_{i+1} inside some symplecton S_i on x. Since all symplecta have rank at least 3, and all lines are thick, by Theorem 3.1 each $\mathrm{w}(a_i, a_{i+1})$ is homotopic inside S_i to the walk (a_i, p, a_{i+1}). Therefore C is homotopic to the walk $(a_0, p, a_1) \circ (a_1, p, a_2) \circ \cdots \circ (a_{n-1}, p, a_n)$, which is contractible.

(ii) Suppose $h \in H$. In this case the graph $\mathcal{S}(h) - H$ can have several connected components.

As was observed earlier, axioms (H0) and (H1) imply that all point residues of Γ are strong parapolar space. Therefore, since all lines are thick, Corollary 3.3 shows that the graph $\Delta|(h^\perp - H)$ is connected. It follows that there is exactly one connected component of $\mathcal{S}(h) - H$ intersecting $h^\perp - H$ nontrivially.

No two distinct symplecta on h can share a point at distance two from h. Therefore all the other connected components of $\mathcal{S}(h) - H$ (if there are any) have the form $S - H$, where S is a symplecton containing h and $h^\perp \cap S \subseteq H$.

Let C be a circular walk in $\mathcal{S}(h) - H$. If C lies entirely in one symplecton S on h, then, since all symplecta of Γ have rank at least 3, and all lines of Γ are thick, the walk C is contractible by Theorem 3.1. Therefore, we can assume that C passes through more than one symplecton. In this case C belongs to the connected component of the graph $\mathcal{S}(h) - H$ containing $h^\perp - H$ and has the form $\mathrm{w}(a_0, a_1) \circ \mathrm{w}(a_1, a_2) \circ \cdots \circ \mathrm{w}(a_{n-1}, a_n)$, where $a_n = a_0$ and, for every $i \in \{0, \ldots, n-1\}$, $a_i \in h^\perp$ and $\mathrm{w}(a_i, a_{i+1})$ is a walk from a_i to a_{i+1} in a symplecton S_i on h, as in case (i) above.

Every symplecton S_i is a nondegenerate polar space of rank at least 3 with thick lines, therefore the residue $\mathrm{Res}_{S_i}(h)$ is a nondegenerate polar space of rank at least 2 with thick lines. So, by Theorem 3.1 $\mathrm{Res}_{S_i}(h)$ minus its hyperplane, consisting of the lines of Γ on h that lie in the subspace $S_i \cap H$, is connected. This implies that the graph $\Delta|(h^{\perp} \cap S_i - H)$ is connected.

For every $i \in \{0, \dots, n-1\}$, let $w'(a_i, a_{i+1})$ be a walk from a_i to a_{i+1} in $h^{\perp} \cap S_i - H$. By Theorem 3.1 applied to each S_i, the walk C is homotopic to the walk $C' = w'(a_0, a_1) \circ w'(a_1, a_2) \circ \cdots \circ w'(a_{n-1}, a_n = a_0)$. The walk C' is contractible in $\Delta - H$ by Lemma 2.6. □

5.2. *Contractibility of certain "pentagons" in $\Delta_2^*(p) - H$*

The definition of walks $\mathrm{P}(a, b)$ can be found in Section 3.3.

Lemma 5.2. *Suppose that the hypothesis of Theorem* 2.1 *holds.*

Suppose further that, for every pair of points $a, b \in \mathcal{P} - H$ with $\mathrm{d}_{\Delta}(a, b) \leqslant 2$, a walk $\mathrm{P}(a, b)$ has been chosen.

Let p, x, and y be points in $\mathcal{P} - H$, such that $\{x, y\}$ is an edge, and both $\{p, x\}$ and $\{p, y\}$ are special pairs. Then the circular walk $C = \mathrm{P}(p, x) \circ (x, y) \circ \mathrm{P}(y, p)$ is contractible in $\Delta - H$.

Proof. Let $\{a\} = x^{\perp} \cap p^{\perp}$ and $\{b\} = y^{\perp} \cap p^{\perp}$. If $a = b$, then C lies entirely inside $a^{\perp} - H$. Therefore the walk C is contractible by Lemma 2.6.

Assume $a \neq b$. In this case by Lemma 3.4 a is collinear with b. Let $S_1 = \mathrm{S}(a, x, y, b)$ be the symplecton on $\{a, x, y, b\}$. Note that $p \notin S_1$ and $p^{\perp} \cap S_1 = \langle a, b \rangle$, since otherwise the pairs $\{x, p\}$ and $\{y, p\}$ would not be special. We consider four cases separately.

Case 1. The line $\langle a, b \rangle$ is in H and it contains the radical r_1 of the hyperplane $S_1 \cap H$ of S_1. In particular, $S_1 \cap H$ is a degenerate hyperplane of S_1. Then, since a is collinear with $x \notin H$ and b is collinear with $y \notin H$, we see that $r_1 \neq a, b$.

Let π be any plane on $\langle a, b \rangle$ inside S_1 and let $\pi' = \langle a, b, p \rangle$. Since $\{p, x\}$ and $\{p, y\}$ are special, we have $\pi \nsubseteq p^{\perp}$, therefore π and π' lie in a unique symplecton. Let $S_2 = \mathrm{S}(\pi, \pi')$.

Let r_2 denote the radical of the hyperplane $S_2 \cap H$ of S_2, if there is one. Since all lines are thick, the plane π contains a point z such that both $r_1, r_2 \notin \langle a, z \rangle$ and $r_1, r_2 \notin \langle b, z \rangle$. We are going to show that $\mathrm{P}(x, p)$ and $\mathrm{P}(y, p)$ are homotopic to some walks X and Y lying entirely in $\mathcal{S}(z) - H$.

We construct X first. Since $\mathrm{Res}_{S_1}(a)$ is nondegenerate polar space, its hyperplane determined by H has at most one deep point, and this deep point is the line $\langle a, r_1 \rangle$. Since $r_1 \notin \langle a, z \rangle$, we have $\langle a, z \rangle \neq \langle a, r_1 \rangle$, therefore there is a plane π_1 on $\langle a, z \rangle$ inside S_1 not contained in H. Similarly, there is a plane π_2 on $\langle a, z \rangle$ inside S_2 not contained in H. If the space $\langle \pi_1, \pi_2 \rangle$ is not singular, then let S_3 be the symplecton $\mathrm{S}(\pi_1, \pi_2)$; if the space $\langle \pi_1, \pi_2 \rangle$ is singular, then let $S_3 = \langle \pi_1, \pi_2 \rangle$. We have $\langle a, z \rangle \subseteq S_1 \cap S_2 \cap S_3$. The walk X will be a concatenation of three walks – X_1, X_2, and X_3 – which lie in the subspaces S_1, S_2, and S_3 respectively.

Pick points $u \in \pi_1 - H$ and $v \in \pi_2 - H$. Since all symplecta of Γ have thick lines, by Corollary 3.3 the graph $\Delta|(S_i - H)$ is connected for all $i \in \{1, 2, 3\}$. Let $X_1 = \mathrm{P}(x, u)$, $X_2 = \mathrm{P}(v, p)$, and $X_3 = \mathrm{P}(u, v)$. Then, for all $i \in \{1, 2, 3\}$, the walk X_i lies in $S_i - H$. We now put $X = X_1 \circ X_3 \circ X_2$. Since $\langle a, z \rangle \subseteq S_1 \cap S_2 \cap S_3$, the walk X lies in $(\mathcal{S}(a) - H) \cap (\mathcal{S}(z) - H)$. In particular, since $\mathrm{P}(x, p)$ and X are both walks in $\mathcal{S}(a) - H$, they are homotopic to each other in $\Delta - H$ by Corollary 5.1.

By symmetry, there is a walk Y from y to p lying inside $(\mathcal{S}(b) - H) \cap (\mathcal{S}(z) - H)$ and homotopic to $\mathrm{P}(y, p)$. Now let $C' = X^{-1} \circ (x, y) \circ Y$. Then the walk C is homotopic to C' in $\Delta - H$. By Corollary 5.1 applied to $\mathcal{S}(z) - H$, the walk C' is contractible in $\Delta - H$.

Case 2. The line $\langle a, b \rangle$ is in H but it does not contain the radical r_1 of the hyperplane $S_1 \cap H$ of S_1. In this case the line $\langle a, b \rangle$ is not the deep point of the hyperplane of $\mathrm{Res}_{S_1}(a)$ determined by H, therefore there is a plane π on $\langle a, b \rangle$ inside S_1 which is not in H. That is, $\pi \cap H = \langle a, b \rangle$. As in Case 1, let $\pi' = \langle a, b, p \rangle$ and let $S_2 = \mathrm{S}(\pi, \pi')$. Let z be any point in $\pi - \langle a, b \rangle$. Again, we are going to replace $\mathrm{P}(x, p)$ and $\mathrm{P}(y, p)$ with two walks X and Y homotopic to them in $\Delta - H$.

Let $X_1 = \mathrm{P}(x, z)$ and let $X_2 = \mathrm{P}(z, p)$. Then X_1 lies in $S_1 - H$ and X_2 lies in $S_2 - H$. Let $X = X_1 \circ X_2$. By Corollary 5.1 applied to $\mathcal{S}(a) - H$ the walk $\mathrm{P}(x, p)$ is homotopic to X in $\Delta - H$.

Similarly, let $Y = Y_1 \circ Y_2$, where $Y_1 = \mathrm{P}(y, z)$ and $Y_2 = \mathrm{P}(z, p)$. By Corollary 5.1 applied to $\mathcal{S}(b) - H$ the walks $\mathrm{P}(y, p)$ and Y are homotopic in $\Delta - H$.

Let $C' = X^{-1} \circ (x, y) \circ Y$. Then the walk C is homotopic to C'. Since the walk C' lies in $\mathcal{S}(z) - H$, it is contractible in $\Delta - H$ by Corollary 5.1.

Case 3. The line $\langle a, b\rangle$ is not in H but a or b, say b, is in H. In this case let $X = (x, a, p)$ and let $Y = \mathrm{P}(y, a) \circ (a, p)$. By Corollary 5.1 $\mathrm{P}(x, p)$ is homotopic to X, since they both lie in $\mathcal{S}(a) - H$, and $\mathrm{P}(y, p)$ is homotopic to Y, since both lie in $\mathcal{S}(b) - H$. Therefore the walk C is homotopic to $C' = X^{-1} \circ (x, y) \circ Y$, which is contractible by Corollary 5.1 applied to $\mathcal{S}(b) - H$ (or just by Theorem 3.1 applied to S_1).

Case 4. The line $\langle a, b\rangle$ is not in H and $\langle a, b\rangle \cap H = h \neq a, b$. Then let $X = (x, a, p)$ and let $Y = (y, b, p)$. By Corollary 5.1 applied to $\mathcal{S}(a) - H$ the walk $\mathrm{P}(x, p)$ is homotopic to X in $\Delta - H$. Similarly, the walk $\mathrm{P}(y, p)$ is homotopic to the walk Y in $\Delta - H$. Let $C' = X^{-1} \circ (x, y) \circ Y$. Then the walk C is homotopic to C' in $\Delta - H$. The walk C' is contractible in $\Delta - H$ by Theorem 3.1 since it breaks up into a square and a triangle. This completes the proof of the lemma. □

5.3. Proof of Proposition 2.4

Proof of Proposition 2.4. First, for every pair of points $a, b \in \mathcal{P}$ with $\mathrm{d}_\Delta(a, b) \leqslant 2$, we choose a walk $\mathrm{P}(a, b)$.

Let $C = (x_0, x_1, \dots, x_n)$ be a circular walk in $\Delta_2^*(p) - H$. For every $i = 1, \dots, n$, let $C_i = \mathrm{P}(p, x_i) \circ (x_i, x_{i+1}) \circ \mathrm{P}(x_{i+1}, p)$. To prove contractibility of C in $\Delta - H$ it suffices to show that each C_i is contractible in $\Delta - H$. We consider several cases.

Case 1. $x_i, x_{i+1} \in p^\perp$. Then C_i is contractible, since in this case C_i is a triangle.

Case 2. $x_i \in p^\perp$ but $x_{i+1} \notin p^\perp$. Then $C_i = (p, x_i, x_{i+1}) \circ \mathrm{P}(x_{i+1}, p)$. If the pair $\{x_{i+1}, p\}$ is symplectic, then $x_i \in \mathrm{S}(x_{i+1}, p)$ and the entire walk C_i lies in $\mathrm{S}(x_{i+1}, p) - H$. Therefore, C_i is contractible by Theorem 3.1.

Suppose that the pair $\{x_{i+1}, p\}$ is special. Then $x_{i+1}^\perp \cap p^\perp = \{x_i\}$ and the walk C_i lies in $x_i^\perp - H$. Therefore, C_i is contractible.

Case 3. $x_i, x_{i+1} \notin p^\perp$ and both $\{x_i, p\}$ and $\{x_{i+1}, p\}$ are special. This case is precisely Lemma 5.2.

Case 4. $x_i, x_{i+1} \notin p^\perp$ and $\{x_i, p\}$ is special but $\{x_{i+1}, p\}$ is symplectic. Let $a_i \in \mathcal{P}$ be such that $\{a_i\} = p^\perp \cap x_i^\perp$. Since the intersection $x_i^\perp \cap \mathrm{S}(x_{i+1}, p)$ contains x_{i+1}, by the axiom (H1) it contains a line L on x_{i+1}. Since $\mathrm{S}(x_{i+1}, p)$ is a polar space, the intersection $p^\perp \cap L$ is nonempty. Therefore, $p^\perp \cap L = \{a_i\}$, in particular $a_i \in \mathrm{S}(x_{i+1}, p)$. The graph $\Delta|(a_i^\perp)$ is contained in $\mathcal{S}(a_i)$, therefore the walk $\mathrm{P}(p, x_i) \circ (x_i, x_{i+1})$ lies in $\mathrm{S}(a_i)$. It follows that the entire walk C_i lies in $\mathcal{S}(a_i) - H$, therefore C_i is contractible in $\Delta - H$ by Corollary 5.1.

Case 5. $x_i, x_{i+1} \notin p^\perp$ and both $\{x_i, p\}$ and $\{x_{i+1}, p\}$ are symplectic. If $\mathrm{S}(x_i, p) = \mathrm{S}(x_{i+1}, p)$, then C_i lies in $\mathrm{S}(x_i, p)$ and is contractible by Theorem 3.1. If $\mathrm{S}(x_i, p) \neq \mathrm{S}(x_{i+1}, p)$, then $x_i^\perp \cap \mathrm{S}(x_{i+1}, p)$ contains a line L on x_{i+1}. The intersection $p^\perp \cap L$ contains a point a distinct from p, which must belong to both $\mathrm{S}(x_i, p)$ and $\mathrm{S}(x_{i+1}, p)$. Since the graph $\Delta|(a^\perp)$ is contained in $\mathcal{S}(a)$, the walk (x_i, x_{i+1}) lies in $\mathcal{S}(a)$. Therefore, the entire walk C_i lies in $\mathcal{S}(a) - H$ and is contractible in $\Delta - H$ by Corollary 5.1. This completes the proof of the present proposition. □

6. Proof of Proposition 2.5

In Sections 6.1 and 6.2 we prove contractibility of hexagons of the first and second kind, respectively, in $\Delta - H$. Section 6.3 contains the proof of Proposition 2.5.

6.1. Contractibility of hexagons of the first kind in $\Delta - H$

The two lemmas in this subsection apply to more general parapolar spaces than hexagonic geometries. In particular, Lemma 6.1 will be used again in Section 7 for local geometries of hexagonic geometries.

Lemma 6.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a parapolar space and let Δ be the point-collinearity graph of Γ. Suppose that Γ has thick lines and satisfies conditions* (H5) *and* (C1)*.*

Let H be a hyperplane of Γ and suppose that (C3) *holds. Then all hexagons of the first kind in $\Delta - H$ are contractible.*

Proof. Suppose $C = \mathrm{w}(p,a) \circ (a,x,a') \circ \mathrm{w}(a',p)$ is a hexagon of the first kind in $\Delta - H$.

Let $X = x^{\perp} \cap \Delta_2^*(p)$. Then X is a subspace of Γ isomorphic to $\mathrm{Res}_{\Gamma}(x)$. Since Γ has thick lines and satisfies condition (C1) (and therefore also (C0)), the space $\mathrm{Res}_{\Gamma}(x)$ is a strong parapolar space with thick lines. The space $X \cap H$ is a hyperplane of X, therefore by Corollary 3.3 the graph $\Delta|(X - H)$ is connected.

Let $A = (a_0, a_1, \dots, a_n)$, where $a_0 = a$ and $a_n = a'$, be a walk from a to a' in $X - H$. Then C is homotopic to the walk $C' = \mathrm{w}(p,a) \circ A \circ \mathrm{w}(a',p)$ which lies in $\Delta_2^*(p) - H$ and, therefore, is contractible by condition (C3). □

The following lemma will be used in the proof of Proposition 2.5.

Lemma 6.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a parapolar space and let Δ be the point-collinearity graph of Γ. Suppose that Γ has thick lines and satisfies conditions* (H5) *and* (C1)*.*

Let H be a hyperplane of Γ. Suppose $a, b, p \in \mathcal{P} - H$ are such that $\mathrm{d}_{\Delta}(p,a) = 3 = \mathrm{d}_{\Delta}(p,b)$ and there exist points $a', b' \in p^{\perp} - H$ with $\mathrm{d}_{\Delta}(a,a') = 2 = \mathrm{d}_{\Delta}(b,b')$.

Then there exists a line L on p, such that for some points $x, y \in L - H$, not necessarily distinct, we have $\mathrm{d}_{\Delta}(a,x) = 2$ and $\mathrm{d}_{\Delta}(b,y) = 2$.

Proof. Let $A = \Delta_2^*(a) \cap p^{\perp}$. Then A is a subspace of Γ and there is a natural isomorphism ϕ_a of A onto $\mathrm{Res}_{\Gamma}(p)$ sending every $u \in A$ to the line $\langle p, u \rangle$ of Γ. Let $B = \Delta_2^*(b) \cap p^{\perp}$. Then B is a subspace of Γ and we let ϕ_b denote a similar isomorphism of B onto $\mathrm{Res}_{\Gamma}(p)$.

Let $H_a = A \cap H$. Then H_a is a subspace of A, and it is not all of A, since $a' \in A - H_a$. Therefore H_a is a hyperplane of A. Similarly, let $H_b = B \cap H$. Then H_b is a hyperplane of B. The subspaces H_a and H_b of Γ determine via ϕ_a and ϕ_b two, not necessarily distinct, hyperplanes of $\mathrm{Res}_{\Gamma}(p)$.

Since Γ is a parapolar space with thick lines satisfying condition (C1), and therefore (C0), the space $\mathrm{Res}_{\Gamma}(p)$ is a strong parapolar space with thick lines. Therefore, by Theorem 3.2 $\mathrm{Res}_{\Gamma}(p)$ is not the union of two hyperplanes $\phi_a(H_a)$ and $\phi_b(H_b)$. It follows that there exists a line L on p, such that $L \cap H_a = \emptyset$ and $L \cap H_b = \emptyset$. Let $x, y \in \mathcal{P}$ be such that $\{x\} = \Delta_2^*(a) \cap L$ and $\{y\} = \Delta_2^*(b) \cap L$. Then x and y both lie in $\mathcal{P} - H$. □

6.2. Contractibility of hexagons of the second kind in $\Delta - H$

Suppose the hypothesis of Theorem 2.1 holds. The purpose of this subsection is to prove Corollary 6.4 that states that all hexagons of the second kind in $\Delta - H$ are contractible.

Lemma 6.3. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a hexagonic geometry and let Δ be the point-collinearity graph of Γ. Suppose Γ has thick lines.*

Let H be a hyperplane of Γ. Suppose that $\mathrm{w}(x,u) \circ (u,v) \circ \mathrm{w}(v,y) \circ (y,x)$ *is a hexagon of the second kind in $\Delta - H$.*

Then there exists a walk (x,a,b,v) in Δ with $a, b \in \mathcal{P} - H$.

Proof. Let $X = \Delta_2^*(v) \cap x^{\perp}$ and let $V = \Delta_2^*(x) \cap v^{\perp}$. Then X is a subspace of Γ isomorphic to $\mathrm{Res}_{\Gamma}(x)$, and V is a subspace of Γ isomorphic to $\mathrm{Res}_{\Gamma}(v)$. In particular, both X and V are strong parapolar spaces with thick lines.

By Corollary 3.5 the map $\phi : X \to V$, that takes every $z \in X$ to the unique point $w \in V$ such that $\{w\} = z^{\perp} \cap v^{\perp}$, determines an isomorphism of the induced subspaces $\Gamma|X \to \Gamma|V$, which we also denote ϕ.

Let $H_x = X \cap H$ and let $H_v = V \cap H$. Then H_x and $\phi^{-1}(H_v)$ are hyperplanes of X.

By Theorem 3.2 the subspace X of Γ is not the union of two hyperplanes H_x and $\phi^{-1}(H_v)$. Therefore, there exists a point $a \in X - (H_x \cup \phi^{-1}(H_v))$. Let $b = \phi(a)$. Then $b \in V - H_v$, therefore (x,a,b,v) is a walk in Δ avoiding H. □

Corollary 6.4. *Suppose that the hypothesis of Theorem* 2.1 *holds. Then condition* (C4) *holds.*

Proof. Suppose $C = \mathrm{w}(x, u) \circ (u, v) \circ \mathrm{w}(v, y) \circ (y, x)$ is a hexagon of the second kind in $\Delta - H$. Let (x, a, b, v) be the walk from x to v in $\Delta - H$ constructed in Lemma 6.3. Let $C_1 = \mathrm{w}(x, u) \circ (u, v, b, a, x)$ and let $C_2 = \mathrm{w}(v, y) \circ (y, x, a, b, v)$. If C_1 and C_2 are contractible in $\Delta - H$, then so is C.

By Proposition 2.4 Γ satisfies (C3). The walks C_1 and C_2 are hexagons of the first kind, therefore they are contractible in $\Delta - H$ by Lemma 6.1. $\square$

6.3. Proof of Proposition 2.5

Proposition 2.5 will be an immediate corollary of the following lemma; this lemma will be used again in Section 7.

Lemma 6.5. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a parapolar space and let Δ denote the point-collinearity graph of Γ.*

Suppose that Γ has symplectic rank at least three, has thick lines, and satisfies conditions (H5), (C1), *and* (C2).

Let H be a hyperplane of Γ. Suppose that conditions (C3) *and* (C4) *hold. Then*

(i) *the graph $\Delta - H$ is connected and*
(ii) *for every $p \in \mathcal{P} - H$, the subgraph $\Delta|(\Delta_2^*(p) - H)$ of $\Delta - H$ controls homotopy in $\Delta - H$.*

Proof. Statement (i) is immediate from Proposition 4.2. It remains to prove (ii).

Let p be a point in $\mathcal{P} - H$. Let $C = (x_0, x_1, \dots, x_n)$ be a walk in $\Delta - H$, such that $x_0, x_n \in \Delta_2^*(p) - H$. We can assume that x_0 and x_n are the only two vertices of C lying in $\Delta_2^*(p)$ and that, for every i, the point x_i is not collinear with x_{i+2} – any other walk is homotopic to a concatenation of walks with these properties.

By Lemma 6.1 we can assume that C has length at least 3. We need to show that C is homotopic in $\Delta - H$ to a walk C' from x_0 to x_n lying entirely in $\Delta_2^*(p) - H$.

First, for every pairs of points $a, b \in \mathcal{P} - H$ with $\mathrm{d}_\Delta(a, b) \leqslant 2$, we choose a walk $\mathrm{P}(a, b)$, as in Section 3.3. Then, for every $i \in \{0, \dots, n\}$, we define walks $\mathrm{P}^+(x_i, p)$ and $\mathrm{P}^-(x_i, p)$ as follows.

(1) Suppose $\{x_i, x_{i+1}\} \subseteq \mathcal{P} - \Delta_2^*(p)$. Let $L_i = \langle x_i, x_{i+1}\rangle$ be the line spanned by x_i and x_{i+1}, and let M_i be a line on p opposite to L_i. The line M_i exists by condition (C2). Since all lines are thick and $L_i, M_i \nsubseteq H$ there exist a pair of points $a_i \in L_i - H$ and $b_i \in M_i - H$, such that $\mathrm{d}_\Delta(a_i, b_i) = 2$. We put

$$\mathrm{P}^+(x_i, p) = (x_i, a_i) \circ \mathrm{P}(a_i, b_i) \circ (b_i, p)$$

$$\mathrm{P}^-(x_{i+1}, p) = (x_{i+1}, a_i) \circ \mathrm{P}(a_i, b_i) \circ (b_i, p)$$

(2) For edges $\{x_0, x_1\}$ and $\{x_{n-1}, x_n\}$ we let $a_0 = x_0$, $a_{n-1} = x_n$, $b_0 = p$, $b_{n-1} = p$, and

$$\mathrm{P}^+(x_0, p) = P^-(x_0, p) = \mathrm{P}(x_0, p)$$

$$\mathrm{P}^+(x_n, p) = P^-(x_n, p) = \mathrm{P}(x_n, p)$$

$$\mathrm{P}^-(x_1, p) = (x_1, x_0) \circ \mathrm{P}(x_0, p)$$

$$\mathrm{P}^+(x_{n-1}, p) = (x_{n-1}, x_n) \circ \mathrm{P}(x_0, p)$$

For every $i \in \{1, \dots, n-1\}$, let $C_i = P^-(x_i, p) \circ (P^+(x_i, p))^{-1}$.

Let $C' = \mathrm{P}(x_0, p) \circ \mathrm{P}(p, x_n)$. Then C' is a walk from x_0 to x_n in $\Delta_2^*(p) - H$.

To show that C is homotopic to C' it suffices to show that, for every $i \in \{1, \dots, n-1\}$, the walk C_i is contractible in $\Delta - H$. There are the following three cases to consider, depending on wether a_{i-1} and a_i are in $\Delta_2^*(p)$ or not.

Case 1. $a_{i-1}, a_i \in \Delta_2^*(p)$. Then $b_{i-1} = b_i = p$ and C_i is contractible by Lemma 6.1.

Case 2. $a_{i-1} \in \Delta_2^*(p)$ and $a_i \in \mathcal{P} - \Delta_2^*(p)$ (the case when $a_{i-1} \in \mathcal{P} - \Delta_2^*(p)$ and $a_i \in \Delta_2^*(p)$ is similar). If $x_i = a_i$, then C_i is a hexagon of the second kind and is contractible by condition (C4).

Suppose that $x_i \neq a_i$. In this case $\mathrm{d}_\triangle(x_i, b_i) = 3$. By Lemma 6.2 (with x_i in the role of p, and p and b_i in the role of a and b) there exists a line L on x_i, such that the points $\{u\} = \Delta_2^*(p) \cap L$ and $\{v\} = \Delta_2^*(b_i) \cap L$ are not in H. Then C_i breaks up into three walks

$$A_1 = \mathrm{P}(p, a_{i-1}) \circ (a_{i-1}, x_i, u) \circ \mathrm{P}(u, p)$$

$$A_2 = \mathrm{P}(p, u) \circ (u, v) \circ \mathrm{P}(v, b_i) \circ (b_i, p)$$

$$A_3 = \mathrm{P}(b_i, v) \circ (v, x_i, a_i) \circ \mathrm{P}(a_i, b_i)$$

and the triangle (x_i, u, v). The walks A_1 and A_3 are contractible by Lemma 6.1. If $u = v$, then the circuit A_2 lies entirely in $\Delta_2^*(u) - H$ and is contractible by condition (C3). If $u \neq v$, then $\mathrm{d}_\triangle(b_i, u) = 3$, the walk A_2 is a hexagon of the second kind and, therefore, is contractible by condition (C4).

Case 3. $a_{i-1}, a_i \in \mathcal{P} - \Delta_2^*(p)$. Then by Lemma 6.2 there exists a line L on p, such that the points $\{x\} = L \cap \Delta_2^*(a_{i-1})$ and $\{y\} = L \cap \Delta_2^*(a_i)$ are not in H.

The two circular walks $A_1 = (p, b_{i-1}) \circ \mathrm{P}(b_{i-1}, a_{i-1}) \circ \mathrm{P}(a_{i-1}, x) \circ (x, p)$ and $A_3 = (p, b_i) \circ \mathrm{P}(b_i, a_i) \circ \mathrm{P}(a_i, y) \circ (y, p)$ are contractible by Lemma 6.1.

Let $A_2 = \mathrm{P}(x, a_{i-1}) \circ (a_{i-1}, x_i, a_i) \circ \mathrm{P}(a_i, y) \circ (y, x)$ be the "middle" circular walk. Suppose $x = y$. Then either $\mathrm{d}_\triangle(x, x_i) = 2$, A_2 lies in $\Delta_2^*(x)$ and is contractible by condition (C3), or $\mathrm{d}_\triangle(x, x_i) = 3$ and A_2 is contractible by Lemma 6.1.

Suppose $x \neq y$. Since the line L contains p, and $\mathrm{d}_\triangle(p, x_i) = 3$, the points x and y cannot be both at distance 2 from x_i. If exactly one of them, say x, is at distance 2 from x_i, then A_2 decomposes into walks $A_2' = \mathrm{P}(x, a_{i-1}) \circ (a_{i-1}, x_i) \circ \mathrm{P}(x_i, x)$ and $A_2'' = \mathrm{P}(x, x_i) \circ (x_i, a_i) \circ \mathrm{P}(a_i, y) \circ (y, x)$, which are contractible by (C3) and (C4) respectively.

Thus we can assume that $x \neq y$, and $\mathrm{d}_\triangle(x, x_i) = 3$ and $\mathrm{d}_\triangle(y, x_i) = 3$. By Lemma 6.2 applied to x, y, and x_i there exists a line M on x_i, such that the points $\{u\} = \Delta_2^*(x) \cap M$ and $\{v\} = \Delta_2^*(y) \cap M$ are not in H. The circular walks $A_2' := \mathrm{P}(x, a_{i-1}) \circ (a_{i-1}, x_i, u) \circ \mathrm{P}(u, x)$ and $A_2''' := \mathrm{P}(y, v) \circ (v, x_i, a_i) \circ \mathrm{P}(a_i, y)$ are contractible by Lemma 6.1. The walk $A_2'' = \mathrm{P}(x, u) \circ (u, v) \circ \mathrm{P}(v, y) \circ (y, x)$ is contractible by (C3) if $u = v$ and by (C4) if $u \neq v$. This completes the proof of the lemma. □

Proof of Proposition 2.5. Suppose that Γ is a hexagonic geometry that satisfies condition (A). Then Γ has thick lines and condition (C1) follows from the axioms (H0) and (H1).

By Lemma 1.1 condition (H5) holds. By Lemma 4.1 Γ satisfies condition (C2).

Let H be a hyperplane of Γ. Then the pair Γ and H satisfy condition (C3) by Proposition 2.4 and condition (C4) by Corollary 6.4.

All the conditions of Lemma 6.5 are met, therefore we obtain the conclusion of Proposition 2.5. □

7. Local geometries of hexagonic geometries

The purpose of this section is to prove Lemma 2.6. To achieve this, we consider a class of strong parapolar spaces that includes all point residues of hexagonic geometries. This class is described in Section 7.1.

In Sections 7.2 and 7.3 we prove Propositions 7.2 and 7.3 (stated in Section 7.1). These two propositions form a basis for the proof of Lemma 2.6 for hexagonic geometries of symplectic rank at least four and for hexagonic geometries of symplectic rank three, respectively.

The proof of Lemma 2.6 is given in Section 7.4.

7.1. A class of strong parapolar spaces

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry and let $\triangle$ be the point-collinearity graph of Γ. Consider the following axioms.

(L0) Γ is a strong parapolar space.
(L1) For every point $x \in \mathcal{P}$ and every symplecton S of Γ, $x^\perp \cap S \neq \emptyset$.
(L2) For every point $p \in \mathcal{P}$, the set $\Delta_2^*(p) = \{q \in \mathcal{P} \mid \mathrm{d}_\triangle(p, q) \leqslant 2\}$ is a hyperplane of Γ.

Theorem 7.1. *(See Theorem 21 of [9].) Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry and let Δ be the point-collinearity graph of Γ. Suppose that Γ satisfies axioms* (L0)–(L2)*. Then the following holds.*

(L3) *Every geodesic of length* 2 *in Δ is extendable to a geodesic of length* 3.

Let Γ be a point-line geometry that satisfies axioms (L0)–(L2). Suppose that Γ has thick lines. Then it was shown in [9] (Theorems 30, 34, and 35; these theorems do not use the restriction (P3) of [9] on the singular rank of Γ) that Γ satisfies one of the following two conditions.

(R1) All symplecta of Γ have rank at least 3.
(R2) The geometry Γ is a dual polar space of rank 3 or a product geometry $L \times \mathbb{P}$, where L is a line and $\mathbb{P}$ is a nondegenerate polar space of rank at least two.

Suppose Γ is a point residue of a hexagonic geometry. Then Γ satisfies conditions (L0)–(L2) (see Section 1.2). Therefore Lemma 2.6 will be an immediate corollary of the following two propositions and Theorem 3.1.

Proposition 7.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry and let Δ denote the point-collinearity graph of Γ. Suppose that Γ satisfies conditions* (L0)–(L2) *and* (R1)*, and has thick lines.*

Let H be a hyperplane of Γ. Then the graph $\Delta - H$ is simply connected.

Proposition 7.3. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry and let Δ denote the point-collinearity graph of Γ. Suppose that Γ satisfies condition* (R2) *and the following condition* $(*)$.

$(*)$ *Every line of Γ has at least* 4 *points. Furthermore, Γ is not the product geometry $L \times L \times L$, where L is a line with exactly* 4 *points.*

Let H be a hyperplane of Γ, and let $\mathcal{C}$ be the set of all backtracks and all circular walks in $\Delta - H$ that lie inside symplecta of Γ.

Then the graph $\Delta - H$ is $\mathcal{C}$-simply connected.

Proposition 7.2 will be proved in Section 7.2 and Proposition 7.3 will be proved in Section 7.3. The proofs of both use Lemmas 7.4–7.7 that we state below.

Proofs of Lemmas 7.4–7.6 can be found in [9]. These proofs do not use the restriction (P3) of [9] on the singular rank of Γ.

Lemma 7.4. *(See Theorem 22 of [9].) Let Γ be a point-line geometry satisfying axioms* (L0)–(L2)*, and let Δ be the point-collinearity graph of Γ.*

Suppose that Γ has thick lines. Then every plane of Γ lies in a symplecton.

Lemma 7.5. *(See Theorem 25 of [9].) Let Γ be a point-line geometry satisfying axioms* (L0)–(L2)*, and let Δ be the point-collinearity graph of Γ.*

Suppose that Γ has thick lines. Then any two symplecta of Γ intersecting at a point already intersect at least at a line.

To state Lemma 7.6 we need the notion of strongly gated subgraph.

Let $G = (V, E)$ be a graph. Suppose $F = (V', E')$ is a subgraph of V. Let $x \in V$, and suppose that there is a vertex $g \in V'$ such that, for every $y \in V'$, we have $\mathrm{d}_G(x, y) = \mathrm{d}_G(x, g) + \mathrm{d}_F(g, y)$. Then we say that the subgraph F is *strongly gated* in G *with respect to* x with gate g, and we write $g = \mathrm{gate}_F(x)$. If F is strongly gated with respect to every vertex of G, then we say that F is *strongly gated* in G.

Suppose X is a subset of V. If the subgraph $G|X$ of G is strongly gated in G with respect to a vertex $x \in V$, then we say that the set X is strongly gated in G with respect to x. Similarly, X is strongly gated in G if the subgraph of $G|X$ of G is strongly gated in G.

Lemma 7.6. *(See Theorem 26 of [9].) Let Γ be a point-line geometry satisfying axioms* (L0)–(L2)*, and let Δ be the point-collinearity graph of Γ.*

Suppose that Γ has thick lines.

Let S be a symplecton of Γ, and suppose x is a point of Γ such that $x^\perp \cap S$ is a single point. Then S is strongly gated in Δ with respect to x with gate g, where $\{g\} = x^\perp \cap S$.

The statement of the next lemma uses the definition of opposite lines and condition (C2). Both of these can be found in Section 3.3, in parts 2 and 3 respectively.

Lemma 7.7. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry satisfying axioms* (L0)–(L2)*. Then Γ satisfies condition* (C2)*.*

Proof. Let $x \in \mathcal{P}$ and let $L \in \mathcal{L}$. Suppose that $y \in L$ is such that $\mathrm{d}_\Delta(x, y) = 3$.

By the axiom (L2) there is a unique point $u \in \mathcal{P}$ such that $\{u\} = L \cap \Delta_2^*(x)$. By (L3) of Theorem 7.1 there exists a point $v \in x^\perp$, such that $\mathrm{d}_\Delta(u, v) = 3$.

Let $M = \langle x, v\rangle$. Since $v \notin \Delta_2^*(u)$, we have $M \not\subseteq \Delta_2^*(u)$. Since $y \notin \Delta_2^*(x)$, we have $L \not\subseteq \Delta_2^*(x)$. Therefore, the lines L and M are opposite in Γ. □

7.2. Local geometries of symplectic rank at least 3

The purpose of this subsection is to prove Proposition 7.2. Geometries in the second column of Table 1 included in Proposition 7.2 are $A_{5,3}$, $D_{6,6}$, and $E_{7,1}$.

Our proof of Proposition 7.2 follows the same pattern as the proof of Theorem 2.1.

Suppose H is a hyperplane of Γ. Let $p \in \mathcal{P} - H$. First, in Proposition 7.10, we show that the subgraph $\Delta|(\Delta_2^*(p) - H)$ of $\Delta - H$ is contractible in $\Delta - H$. Then we use Lemma 6.5 to show that $\Delta_2^*(p) - H$ controls homotopy in $\Delta - H$.

To be able to use Lemma 6.5 we need to show that all hexagons of the second kind in $\Delta - H$ are contractible, that is condition (C4) holds. This is done in Lemma 7.11.

Before we can prove Proposition 7.10, we show that certain "pentagons" in $\Delta(p) - H$ are contractible (Lemmas 7.8 and 7.9).

Lemma 7.8. *Suppose that the hypothesis of Proposition* 7.2 *holds.*

Let x, y, and p be points of Γ, such that $\mathrm{d}_\Delta(x, p) = 2$ and $\mathrm{d}_\Delta(y, p) = 2$. Then there exists a point $q \in p^\perp$ at distance 3 *in Δ from both x and y.*

Proof. By (L3) of Theorem 7.1 there exist points $a, b \in p^\perp$ such that $\mathrm{d}_\Delta(x, a) = 3$ and $\mathrm{d}_\Delta(y, b) = 3$.

Let S be a symplecton containing $\{a, b, p\}$. If $a \notin b^\perp$ then $S = S(a, b)$; if $a \in b^\perp$, then a symplecton S containing a, b, and p exists by Lemma 7.4. Since $S \cap \Delta_3(x) \neq \emptyset$ the intersection $x^\perp \cap S$ cannot contain a line, therefore it consists of a single point. Similarly, $y^\perp \cap S$ consists of a single point.

Let $\{u\} = x^\perp \cap S$ and $\{v\} = y^\perp \cap S$. By Lemma 7.6 S is gated with respect to both x and y with gates u and v respectively. Since $\mathrm{d}_\Delta(x, p) = 2$ and $\mathrm{d}_\Delta(y, p) = 2$, we have $p \in u^\perp \cap v^\perp$.

The residue $\mathrm{Res}_S(p)$ is a nondegenerate polar space of rank at least 2 with thick lines. Therefore by Theorem 3.2 $\mathrm{Res}_S(p)$ is not the union of its two hyperplanes determined by $p^\perp \cap u^\perp \cap S$ and $p^\perp \cap v^\perp \cap S$. This means $(p^\perp - (u^\perp \cup v^\perp)) \cap S \neq \emptyset$.

Let $q \in (p^\perp - (u^\perp \cup v^\perp)) \cap S$. Then $\mathrm{d}_\Delta(x, q) = 3 = \mathrm{d}_\Delta(y, q)$. □

The definition of the walks $\mathrm{P}(a, b)$ can be found in Section 3.3, part 4.

Lemma 7.9. *Suppose that the hypothesis of Proposition* 7.2 *holds.*

Suppose that, for every pair of points $a, b \in \mathcal{P} - H$ with $\mathrm{d}_\Delta(a, b) \leqslant 2$, a walk $\mathrm{P}(a, b)$ has been chosen.

Let p, x and y be three points in $\mathcal{P} - H$, such that $x \in y^\perp$ and $\mathrm{d}_\Delta(x, p) = \mathrm{d}_\Delta(y, p) = 2$. Then the walk $C = \mathrm{P}(p, x) \circ (x, y) \circ \mathrm{P}(y, p)$ is contractible in $\Delta - H$.

Proof. If C lies entirely inside one symplecton, then C is contractible by Theorem 3.1. Therefore we can assume $\mathrm{S}(x,p) \neq \mathrm{S}(y,p)$.

By Lemma 7.5 the intersection $\mathrm{S}(x,p) \cap \mathrm{S}(y,p)$ contains a line L on p. Let z be a point in $x^\perp \cap L$. Since $x \notin \mathrm{S}(y,p)$, by convexity of $\mathrm{S}(y,p)$ we have $x^\perp \cap \mathrm{S}(y,p) \subseteq y^\perp$. Therefore $z \in x^\perp \cap y^\perp \cap p^\perp$. Since $\mathrm{S}(x,p) \neq \mathrm{S}(y,p)$, we have $z \notin \langle x, y\rangle$, therefore, $\langle x, y, z\rangle$ is a plane.

Suppose $z \notin H$. Then C is homotopic to $\mathrm{P}(p,x) \circ (x,z,y) \circ \mathrm{P}(y,p)$, which is contractible by Theorem 3.1 applied to $\mathrm{S}(x,p) - H$ and $\mathrm{S}(y,p) - H$.

Suppose that $z \in H$. Let $S_1 = \mathrm{S}(x,p)$ and $S_2 = \mathrm{S}(y,p)$. By Lemma 7.8 there exists a point $u \in p^\perp$, such that $\mathrm{d}_\Delta(x,u) = 3 = \mathrm{d}_\Delta(y,u)$. By (L3) of Theorem 7.1 we can extend the geodesic (z,p,u) to a geodesic (z,p,u,v) of length 3.

Let $S = \mathrm{S}(p,v)$. Then $z^\perp \cap S = \{p\}$ is a single point, since S contains a point at distance 3 from z.

By Lemma 7.5 the intersections $S \cap S_1$ and $S \cap S_2$ each contain a line on p. Let L_1 be a line in $S \cap S_1$ containing p, and let L_2 be a line in $S \cap S_2$ containing p. If $L_1 = L_2$ then $L_1 \subseteq S_1 \cap S_2$. The set $S_1 \cap S_2$ is a singular subspace containing z and therefore $z^\perp$ contains L_1. This contradicts the fact that $z^\perp \cap S = \{p\}$. Therefore the lines L_1 and L_2 must be distinct.

Let $a \in x^\perp \cap L_1$ and let $b \in y^\perp \cap L_2$. Note that $a \neq b$, because the only common point of L_1 and L_2 is p, and p is not in $x^\perp$ or in $y^\perp$. Since S contains points at distance 3 from both x and y (the point u is distance 3 from both), we have $x^\perp \cap S = \{a\}$ and $y^\perp \cap S = \{b\}$. By Lemma 7.6 every point of S at distance 2 from x lies in $a^\perp$. In particular, $b \in a^\perp$ since (x,y,b) is a walk of length 2 from x to b.

Let S_3 be the symplecton containing (a,x,y,b). Such a symplecton exists and is unique, since $x^\perp \cap S = \{a\}$ and therefore $b \notin x^\perp$. Similarly, $y^\perp \cap S = \{b\}$ and so $a \notin y^\perp$. It follows that $\langle x,a\rangle$ and $\langle y,b\rangle$ are opposite lines of S_3. Also, the lines $\langle x,a\rangle$ and $\langle z,p\rangle$ are opposite inside S_1, since $p \notin x^\perp$ and $a \notin z^\perp$ (the latter is true because $z^\perp \cap S = \{p\}$). Similarly, the lines $\langle y,b\rangle$ and $\langle z,p\rangle$ are opposite in S_2.

We are now ready to show contractibility of C. Recall that we are assuming $z \in H$. We consider three cases.

Case 1. Suppose $a, b \notin H$. Then C is homotopic to (p,a,x,y,b,p), which is homotopic to (p,a,b,p) using a homotopy in S_3. Therefore C is contractible in $\Delta - H$ in this case.

Case 2. Suppose $a, b \in H$. Pick $a_1 \in \langle a,x\rangle$ distinct from a and x, and let $\{b_1\} = a_1^\perp \cap \langle y,b\rangle$, and $\{p_1\} = a_1^\perp \cap \langle z,p\rangle$. Then p_1 and b_1 both lie in $a_1^\perp \cap S_2$, which is a singular space, therefore p_1 is collinear with b_1. Since $\langle a,x\rangle$ is opposite to $\langle z,p\rangle$, and $\langle a,x\rangle$ is opposite to $\langle y,b\rangle$, we have $p_1 \neq z$ and $b_1 \neq b$. Therefore $p_1, b_1 \notin H$. Now the circuit C is homotopic to $(p,p_1,a_1,x,y,b_1,p_1,p)$ to (p,p_1,a_1,b_1,p_1,p) to (p,p_1,p). Therefore C is contractible in $\Delta - H$.

Case 3. Suppose $a \in H$ but $b \notin H$ (the case $a \notin H$ and $b \in H$ is similar). Let $\{b_1\} = \langle p,b\rangle \cap H$ and let $\{y_1\} = b_1^\perp \cap \langle y,z\rangle$. Then C is homotopic to $C' = \mathrm{P}(p,x) \circ (x,y_1) \circ \mathrm{P}(y_1,p)$. The circuit C' is contractible by the same argument as in case (2), using the pairwise opposite lines $\langle a,x\rangle$, $\langle z,p\rangle$ and $\langle y_1,b_1\rangle$. □

Remark. One can check that in the proof of Lemma 7.9 all homotopies are taking place inside symplecta S, such that $p^\perp \cap S$ contains a line. Therefore, we have proved that C is contractible inside the graph $\Delta|(\Delta_2^*(p) - H)$.

Proposition 7.10. *Suppose that the hypothesis of Proposition* 7.2 *holds. Then Γ satisfies condition* (C3).

Proof. Let $p \in \mathcal{P} - H$. We need to show that the subgraph $\Delta|(\Delta_2^*(p) - H)$ of $\Delta - H$ is contractible in $\Delta - H$.

For every pair of points $a, b \in \mathcal{P} - H$ with $\mathrm{d}_\Delta(a,b) \leqslant 2$, choose a walk $\mathrm{P}(a,b)$.

Let $C = (x_0, x_1, \dots, x_n)$ be a circular walk in $\Delta_2^*(p) - H$. For every $i \in \{1, \dots, n\}$, let $C_i = \mathrm{P}(p,x_i) \circ (x_i, x_{i+1}) \circ \mathrm{P}(x_{i+1}, p)$.

To prove contractibility of C it suffices to show that, for every $i \in \{1, \dots, n\}$, the walk C_i is contractible in $\Delta - H$. This is clear if $x_i, x_{i+1} \in p^\perp$, since in this case C_i is contained in $p^\perp$.

Suppose $x_i \in p^\perp$ but $x_{i+1} \notin p^\perp$ (the case $x_{i+1} \in p^\perp$ but $x_i \notin p^\perp$ is similar). Then $x_i \in \mathrm{S}(x_{i+1}, p)$, therefore the entire walk $C_i = (p,x_i) \circ (x_i, x_{i+1}) \circ \mathrm{P}(x_{i+1}, p)$ lies in $\mathrm{S}(x_{i+1}, p)$. By Theorem 3.1 applied to $\mathrm{S}(x_{i+1}, p) - H$ the walk C_i is contractible.

Suppose $x_i, x_{i+1} \notin p^\perp$. Then C_i is contractible in $\Delta - H$ by Lemma 7.9. This completes the proof. $\square$

Remark. By the remark immediately following the proof of Lemma 7.9 the graph $\Delta_2^*(p) - H$ is simply connected as a graph, but we are not going to use this fact.

In view of Proposition 7.10, to prove Proposition 7.2 it remains to show that, for some $p \in \mathcal{P} - H$, the subgraph $\Delta|(\Delta_2^*(p) - H)$ controls homotopy in $\Delta - H$. This will follow from Lemma 6.5, once we show that its hypothesis holds.

The following lemma is the local analogue of Lemma 6.3 and shows that all hexagons of the second kind in $\Delta - H$ are contractible.

Lemma 7.11. *Assume that the hypothesis of Proposition* 7.2 *holds. Then condition* (C4) *is satisfied.*

Proof. Suppose that $C = \mathrm{w}(x, u) \circ (u, v) \circ \mathrm{w}(v, y) \circ (y, x)$ is a hexagon of the second kind in $\Delta - H$. By Proposition 7.10 $\Delta_2^*(x) - H$ and $\Delta_2^*(v) - H$ are contractible, therefore we can assume $\mathrm{w}(x, u) = \mathrm{P}(x, u)$ and $\mathrm{w}(v, y) = \mathrm{P}(v, y)$, where $\mathrm{P}(x, u)$ and $\mathrm{P}(v, y)$ are defined as in Section 3.3.

We consider three cases.

Case 1. Suppose $\mathrm{d}_\Delta(y, u) = 1$. Then $y \in \mathrm{S}(x, u)$ and $u \in \mathrm{S}(y, v)$. By Theorem 3.1 applied to $\mathrm{S}(x, u) - H$ the walk $\mathrm{P}(x, u)$ is homotopic to (x, y, u). Therefore, C is homotopic to the walk $C' = (x, y, u, v) \circ \mathrm{P}(v, y) \circ (y, x)$. The walk C' is contractible by Theorem 3.1 applied to $\mathrm{S}(y, v) - H$.

Case 2. Suppose that $\mathrm{d}_\Delta(u, y) = 2$. Then x, y, u, v, and all vertices of $\mathrm{P}(v, y)$ belong to $\Delta_2^*(y) - H$. Let $S = \mathrm{S}(u, x)$. Since $\mathrm{d}_\Delta(x, u) = \mathrm{d}_\Delta(y, u)$, the point x cannot be the gate of y in S, therefore by Lemma 7.6 the intersection $y^\perp \cap S$ must contain a line L on x. This implies that $S \subseteq \Delta_2^*(y)$. Therefore, C lies in $\Delta_2^*(y) - H$ and is contractible by Proposition 7.10.

Case 3. Suppose that $\mathrm{d}_\Delta(u, y) = 3$. One can show using opposite lines that, in a polar space with thick lines minus a hyperplane, any two points can be connected by a walk of length at most 3. Therefore by Theorem 3.1, replacing if necessary $\mathrm{P}(x, u)$ and $\mathrm{P}(y, v)$ with walks homotopic to them in $\mathrm{S}(x, u) - H$ and $\mathrm{S}(y, v) - H$, we can assume that $\mathrm{P}(x, u)$ and $\mathrm{P}(y, v)$ are walks of length at most 3.

Suppose first that $\mathrm{P}(x, u) = (x, a, u)$ and $\mathrm{P}(v, y) = (v, b, y)$ for some $a, b \in \mathcal{P} - H$. Then the circuit C is a hexagon and the walk (u, a, x, y, b) lies in $\Delta_2^*(x)$. The point v is the only point of C that lies outside $\Delta_2^*(x)$, and v is adjacent to $u, b \in \Delta_2^*(x) - H$. By (L0) and (R1) all point residues of Γ are connected, by (L2) Γ satisfies (H5), and by Proposition 7.10 $\Delta_2^*(x) - H$ is contractible. Therefore the circuit C is contractible by Lemma 6.1.

Suppose now that at least one of the walks $\mathrm{P}(v, y)$ and $\mathrm{P}(x, u)$, say $\mathrm{P}(v, y)$, has length 3, and suppose that it is a shortest possible walk from v to y in $\mathrm{S}(v, y) - H$. Let $a, b \in \mathrm{S}(v, y)$ be such that $\mathrm{P}(v, y) = (v, a, b, y)$.

By the hypothesis of Case 3 we have $\mathrm{d}_\Delta(u, y) = 3$. Therefore $u^\perp \cap \mathrm{S}(v, y)$ cannot contain a line. This implies that $u^\perp \cap \mathrm{S}(v, y) = \{v\}$. Therefore by Lemma 7.6 $\mathrm{S}(v, y)$ is strongly gated with respect to u with gate v.

Similarly, since $\mathrm{d}_\Delta(x, v) = 3$, we have $x^\perp \cap \mathrm{S}(v, y) = \{y\}$. Therefore by Lemma 7.6 $\mathrm{S}(v, y)$ is strongly gated with respect to x with gate y.

Let $S = \mathrm{S}(x, b)$. We have $y \in S$ and by the current hypothesis $\mathrm{d}_\Delta(u, y) = 3$, therefore the symplecton S is strongly gated with respect to u.

Let $g = \mathrm{gate}_S(u)$. Since $\mathrm{d}_\Delta(u, x) = 2$, we have $g \in x^\perp \cap u^\perp$. Since $\mathrm{d}_\Delta(u, b) = 3$, we have $b^\perp \cap \langle g, x\rangle = \{b'\} \neq \{g\}$.

Suppose first that $g \in H$. Then $b' \notin H$, since $\langle g, x\rangle \nsubseteq H$. The walk C is homotopic in $\Delta - H$ to the walk $C' = \mathrm{P}(u, b') \circ (b', b, a, v, u)$. The walk C' is contractible by Lemma 6.1, since it has only one point b outside $\Delta_2^*(u)$.

Suppose now that $g \notin H$, in which case it might happen that $b' \in H$. Then C is homotopic to $C' = (u, g, x, y, b, a, v, u)$. The walk C' lies completely inside $\Delta_2^*(y)$, except for the point u, therefore it is contractible by Lemma 6.1. This completes the proof. $\square$

Proof of Proposition 7.2. First, we claim that Γ satisfies the hypothesis of Lemma 6.5.

The axiom (L2) coincides with condition (H5). By (L0) and (R1) Γ is a strong parapolar space of symplectic rank at least 3, therefore condition (C1) holds. By Lemma 7.7 condition (C2) holds, by Proposition 7.10 condition (C3) holds, and by Lemma 7.11 condition (C4) holds. This proves the claim.

Let $p \in \mathcal{P} - H$. Then by Lemma 6.5 the graph $\Delta - H$ is connected and the subgraph $\Delta|(\Delta_2^*(p) - H)$ controls homotopy in $\Delta - H$. By Proposition 7.10 the graph $\Delta_2^*(p) - H$ is contractible in $\Delta - H$. Combining the two facts and using Lemma 1.2, we obtain that $\Delta - H$ is simply connected. □

7.3. Local geometries containing quads

Our goal in this subsection is to prove Proposition 7.3. The proof of Proposition 7.3 depends on Lemmas 7.12 and 7.13. Lemma 7.12 is the reason for imposing condition ($*$) on the geometry Γ in Proposition 7.3

Lemma 7.12. *Let $\Sigma = (\mathcal{P}_\Sigma, \mathcal{L}_\Sigma)$ be a nondegenerate generalized quadrangle, and let Δ_Σ denote the point-collinearity graph of Σ. Suppose all lines of Σ have at least 4 points, and Σ is not a 4×4 grid.*

Let H_1 and H_2 be two hyperplanes of Σ. Then the graph $\Delta_\Sigma - (H_1 \cup H_2)$ is connected.

Proof. Suppose, first, that $H_1 = H_2$. Then $\Delta_\Sigma - (H_1 \cup H_2)$ is connected by Corollary 3.3.

Suppose that $H_1 \neq H_2$. Let $x, y \in \mathcal{P}_\Sigma - (H_1 \cup H_2)$. We need to show that x and y lie in one connected component of $\Delta_\Sigma - (H_1 \cup H_2)$. This is clear if x and y are collinear.

Suppose that $d_{\Delta_\Sigma}(x, y) = 2$. We consider three cases separately: (1) every line of Σ has at least 5 points; (2) every line of Σ has at least 4 points, and H_1 or H_2 or both contain a line; (3) every line of Σ has at least 4 points, both H_1 and H_2 contain no lines (that is, H_1 and H_2 are ovoids), and Σ is not a grid.

These three cases cover all generalized quadrangles with at least four points on every line, and all choices of hyperplanes H_1 and H_2, except one – the case when Σ is a grid containing a line with exactly 4 points and both hyperplanes are ovoids. Since a grid that has an ovoid must be a square grid, the 4×4 grid is the only exception.

Case 1. Suppose that every line of Σ has at least 5 points. Since Σ is nondegenerate, there exist opposite lines L containing x and M containing y. Then at most 2 points of L and at most 2 points of M are in $H_1 \cup H_2$, therefore there exist a pair of points $x_1 \in L - (H_1 \cup H_2)$ and $y_1 \in M - (H_1 \cup H_2)$, such that (x, x_1, y_1, y) is a walk from x to y in $\Delta_\Sigma - (H_1 \cup H_2)$.

Case 2. Suppose that every line of Σ has at least 4 points, and at least one of the hyperplanes, assume it is H_1, contains a line (that is, H_1 is not an ovoid). Then H_1 is a polar space, its point-collinearity graph is connected, and every point of H_1 lies on a line of Σ contained in H_1.

By Corollary 3.3, $H_1 \nsubseteq H_2$. Therefore, $H_1 - (H_1 \cap H_2) \neq \emptyset$. Let $p \in H_1 - (H_1 \cap H_2)$ and let L be a line on p contained in H_1. Let $x_1, y_1 \in \mathcal{P}_\Sigma$ be such that $x^\perp \cap L = \{x_1\}$ and $y^\perp \cap L = \{y_1\}$.

Suppose first that $x_1 \neq y_1$. Then the lines $\langle x, x_1\rangle$ and $\langle y, y_1\rangle$ are opposite. Since every line of Σ has at least 4 points, $x_1, y_1 \in H_1$, and $x_1 \in y_1^\perp$, there exist a pair of points $u \in \langle x, x_1\rangle - (H_1 \cup H_2)$ and $v \in \langle y, y_1\rangle - (H_1 \cup H_2)$, such that $u \in v^\perp$. Then (x, u, v, y) is a walk from x to y in $\mathcal{P}_\Sigma - H$.

Suppose $x_1 = y_1$. Let $z \in L - H_2$ be distinct from x_1 and from the radical of H_1, if H_1 has a radical. Since $|L| \geqslant 4$, such a point z exists.

Let M be a line on z not contained in H_1. Then M is opposite to both $\langle x, x_1\rangle$ and $\langle y, y_1\rangle$. Therefore, by an argument as in the case $x_1 \neq y_1$, the points of $M - (H_1 \cup H_2)$ lie in one connected component of $\Delta_\Sigma - (H_1 \cup H_2)$ with the points of both $\langle x, x_1\rangle - (H_1 \cup H_2)$ and $\langle y, y_1\rangle - (H_1 \cup H_2)$.

Case 3. Suppose every line of Σ has at least 4 points, Σ is not a grid, and both H_1 and H_2 contain no lines (that is, H_1 and H_2 are ovoids). We can assume that $x^\perp \cap y^\perp \subseteq H_1 \cup H_2$, since otherwise there is nothing to prove.

Suppose first that there exists $z \in (x^\perp \cap y^\perp) \cap (H_1 \cap H_2)$. Let L be a line on x distinct from $\langle x, z\rangle$. Then L and $\langle y, z\rangle$ are opposite, and $\langle y, z\rangle \cap H_1 = \langle y, z\rangle \cap H_2$. Since all lines have at least 4 points, there exist a pair of points $a \in L$ and $b \in \langle y, z\rangle$, such that $a \in b^\perp$ and $a, b \notin H_1 \cup H_2$. Then (x, a, b, y) is a walk from x to y in $\Delta_\Sigma - (H_1 \cup H_2)$.

Suppose now that $(x^\perp \cap y^\perp) \cap (H_1 \cap H_2) = \emptyset$. We can assume without loss of generality that there exists a point $z \in (x^\perp \cap y^\perp) \cap (H_1 - H_2)$.

Let $u, v \in \mathcal{P}_\Sigma$ be such that $\langle x, z\rangle \cap H_2 = \{u\}$ and $\langle y, z\rangle \cap H_2 = \{v\}$. Let L be a line on u distinct from $\langle x, z\rangle$ and let M be a line on v distinct from $\langle y, z\rangle$.

By the hypothesis of Case 3 Σ is not a grid, therefore there exists a line N on z distinct from both $\langle x, z\rangle$ and $\langle y, z\rangle$. Then $L, M, N \nsubseteq H_1 \cup H_2$, since H_1 and H_2 are ovoids.

The line N is opposite to both L and M. Since $L \cap H_2 = \{u\}$ is perped to $N \cap H_1 = \{z\}$, and both L and N contain at least 4 points, there exist two points $l \in L$ and $n \in N$, such that $l \in n^\perp$ and $l, n \notin H_1 \cup H_2$. Therefore $L - (H_1 \cup H_2)$ and $N - (H_1 \cup H_2)$ lie in one connected component of $\Delta_\Sigma - (H_1 \cup H_2)$. Similarly, $M - (H_1 \cup H_2)$ and $N - (H_1 \cup H_2)$ lie in one connected component of $\Delta_\Sigma - (H_1 \cup H_2)$.

By a similar argument $L - (H_1 \cup H_2)$ and $\langle y, z\rangle - (H_1 \cup H_2)$ are in one connected component of $\Delta_\Sigma - (H_1 \cup H_2)$, and $M - (H_1 \cup H_2)$ and $\langle x, z\rangle - (H_1 \cup H_2)$ are in one connected component of $\Delta_\Sigma - (H_1 \cup H_2)$.

Combining the results of the two preceding paragraphs we obtain that x and y belong to one connected component of $\Delta_\Sigma - (H_1 \cup H_2)$. This completes the proof of Case 3 and the proof of the lemma. □

Remark. For a 4×4 grid the conclusion of the above lemma fails – there exist two ovoids H_1 and H_2 such that the graph $\Delta_\Sigma - (H_1 \cup H_2)$ has two connected components.

The next lemma uses the following considerations.

Suppose that Γ is a point-line geometry satisfying condition (R2) and let Δ be the point-collinearity graph of Γ. If Γ is a dual polar space, then all symplecta of Γ are strongly gated in Δ.

Suppose that Γ is the product geometry $L \times \mathbb{P}$. Then all symplecta of Γ that are copies of $\mathbb{P}$ are strongly gated in Δ. If R and S are two symplecta of Γ, $S \cap R = \emptyset$, and R is a copy of $\mathbb{P}$, then S is a copy of $\mathbb{P}$.

Lemma 7.13. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry, and let Δ denote its point-collinearity graph. Suppose that Γ satisfies condition* (R2)*.*

Let R be a symplecton of Γ. If $\Gamma = L \times \mathbb{P}$, then assume further that R is a copy of $\mathbb{P}$ (that is $R = \{(\alpha, \beta) \mid \beta$ ranges through $\mathbb{P}\}$ for some $\alpha \in L$).

Let $\pi : \mathcal{P} \to R$ be the map defined by $\pi(x) = \mathrm{gate}_R(x)$ for every $x \in \mathcal{P}$. Then the following hold.

(i) *π is a morphism of the graph Δ onto its subgraph $\Delta|R$.*
(ii) *For every symplecton S of Γ, such that $S \cap R = \emptyset$, the restriction of π to S is an isomorphism of $\Delta|S$ onto $\Delta|R$.*

Proof. To prove (i), let x and y be two distinct collinear points of Γ. Let $\alpha = \mathrm{d}_\Delta(x, \pi(x))$, $\beta = \mathrm{d}_\Delta(y, \pi(y))$, and $\gamma = \mathrm{d}_\Delta(\pi(x), \pi(y))$. Since $\pi(x) = \mathrm{gate}_R(x)$ and $\pi(y) = \mathrm{gate}_R(y)$, we have $\beta + \gamma \leqslant 1 + \alpha$ and $\alpha + \gamma \leqslant 1 + \beta$. Therefore $\gamma \leqslant 1$. That is $\pi(x) = \pi(y)$ or $\{\pi(x), \pi(y)\}$ is an edge of Δ. This proves (i).

We now prove (ii). Let $p \in \mathcal{P} - R$. By the axiom (L1) we have $p^\perp \cap R \neq \emptyset$. Therefore $\{\pi(p)\} = p^\perp \cap R$.

Let S be a symplecton of Γ such that $S \cap R = \emptyset$. Suppose there exist two distinct points $x, y \in S$ with $\pi(x) = z = \pi(y)$. If x is collinear with y then, since there are no planes on z not inside R, the line $\langle x, y\rangle$ contains z. Then $z \in \langle x, y\rangle \cap R \subseteq S \cap R$, a contradiction with $S \cap R = \emptyset$.

Suppose x is not collinear with y. Then, by convexity of S, we have $z \in x^\perp \cap y^\perp \subseteq S$, a contradiction with $S \cap R = \emptyset$. This shows that π is injective when restricted to S.

By the axiom (L1), for every $x \in R$, we have $x^\perp \cap S \neq \emptyset$. Therefore the restriction of π to S is a surjection of S onto R.

We have shown that π maps the set S bijectively onto the set R. By part (i) π maps an edge of the subgraph $\Delta|S$ onto an edge of the subgraph $\Delta|R$. To show that π is an isomorphism of the point-collinearity graphs of S and R it remains to show that every edge of R is the image of an edge of S.

By hypothesis $S \cap R = \emptyset$. Therefore, if $\Gamma = L \times \mathbb{P}$, then S is a copy of $\mathbb{P}$. This shows that the roles of R and S can be interchanged. Then by part (i) the restriction of π^{-1} to R is a morphism of the point-collinearity graph of R into the point-collinearity graph of S. Therefore π is surjective on edges of $\Delta|S$. $\square$

For ease of reference we state without proof the following simple lemma.

Lemma 7.14. *Let $\Sigma = (\mathcal{P}, \mathcal{L})$ be a point-line space. Suppose that Σ has thick lines.*

Suppose that H_1, H_2 are hyperplanes of Σ. Let $p \in H_1 - (H_1 \cap H_2)$ and suppose that p is not a deep point of H_1, if H_1 has deep points.

Then there exists a point $c \in \mathcal{P} - (H_1 \cup H_2)$ collinear with p.

We can now prove Proposition 7.3.

Proof of Proposition 7.3. Let R be a symplecton of Γ. Assume that R is not contained in H. If $\Gamma = L \times \mathbb{P}$, where L is a line and $\mathbb{P}$ is a nondegenerate polar space, then assume further that R is a copy of $\mathbb{P}$, that is $R = \{(\alpha, \beta) \mid \beta \text{ ranges through } \mathbb{P}\}$ for some $\alpha \in L$.

The symplecton R is strongly gated in Δ, and $|x^{\perp} \cap R| = 1$ for all $x \in \mathcal{P} - R$. Let $\pi : \mathcal{P} \to R$ be the map defined by $\pi(x) = \mathrm{gate}_R(x)$. If $x \in R$, then $\pi(x) = x$. If $x \in \mathcal{P} - R$, then $\pi(x) = u$, where $\{u\} = x^{\perp} \cap R$. By Lemma 7.13(i) π is a morphism of the graph Δ onto its subgraph $\Delta|R$.

By Corollary 3.3 the graph $\Delta - H$ is connected. Therefore to prove that $\Delta - H$ is $\mathcal{C}$-contractible, it suffices to prove the following statement (CH).

(CH) *The subgraph $\Delta|(R - H)$ of $\Delta - H$ controls $\mathcal{C}$-homotopy in $\Delta - H$.*

To prove (CH) we need to show that every walk in Δ that begins and ends in $R - H$ is $\mathcal{C}$-homotopic to a walk lying in $R - H$. The proof consists of Steps 1 and 2 below, but first we need the following definitions.

For a symplecton S, let $H(S) = S \cap H$. If $S \nsubseteq H$, then $H(S)$ is a hyperplane of S. In particular, $H(R)$ is a hyperplane of R. Therefore, $H(R)$ has at most one deep point with respect to R. If it exists, then we denote this point d. That is, d denotes the unique point of $H(R)$ such that $d^{\perp} \cap R \subseteq H(R)$.

Let $H' = \{p \in \mathcal{P} \mid \pi(p) \in H(R)\}$. For a symplecton S of Γ, we let $H'(S)$ denote the set $S \cap H'$.

Step 1. *If C is a walk in $\Delta - (H \cup H')$, that begins and ends in $R - H$, then C is $\mathcal{C}$-homotopic in $\Delta - H$ to a walk contained in $R - H$.*

Let $C = (x_0, x_1, \dots, x_n)$ be a walk in $\Delta - (H \cup H')$ such that $x_0, x_n \in R - H$.

Let $W = (g_0, \dots, g_n)$, where $g_i = \pi(x_i)$ for all $i \in \{0, \dots, n\}$. The geometry Γ is a strong parapolar space that has the property that every singular subspace of Γ lies in a symplecton. Therefore, for every $i \in \{0, \dots, n-1\}$, the circular walk $C_i = (g_i, x_i, x_{i+1}, g_{i+1}, g_i)$ is contained in a symplecton of Γ.

By hypothesis of Step 1, for every i, we have $g_i \in R - H$. Therefore, every walk C_i is contained in $\Delta - H$ and is $\mathcal{C}$-contractible. It follows that C is $\mathcal{C}$-homotopic to a walk contained in $R - H$.

Step 2. *If $C = (x_0, x_1, \dots, x_n)$ is a walk in $\Delta - H$, such that $x_0, x_n \in R - H$ and at least one x_i is in H', then C is $\mathcal{C}$-homotopic in $\Delta - H$ to a walk C_5 in $\Delta - (H \cup H')$.*

In order to construct the walk C_5, we consider the following $\mathcal{C}$-homotopies (T1)–(T5) of the walk C.

(T1) Suppose the walk C contains at least three consecutive points whose gate in R is d. That is, $C = (x_0, \dots, a, h_1, h_2, \dots, h_k, b, \dots, x_n)$, where $\pi(a), \pi(b) \neq d$ and, for $i \in \{1, \dots, k\}$, we have $\pi(h_i) = d$, where $k \geqslant 3$.

Since R contains all planes of Γ on any point of R, there are no planes in Γ on d not in R. Therefore $\langle h_1, d\rangle = \langle h_2, d\rangle = \cdots = \langle h_k, d\rangle$. It follows that, if $h_1 \neq h_k$, then C is homotopic to the walk $C' = (x_0, \dots, a, h_1, h_k, b, \dots, x_n)$ and, if $h_1 = h_k$, then C is homotopic to the walk $C' = (x_0, \dots, a, h_k, b, \dots, x_n)$.

(T2) Suppose the walk C contains two consecutive points whose gate in R is d. That is, $C = (x, \dots, a, h_1, h_2, b, \dots, y)$, where $\pi(h_1) = \pi(h_2) = d$ and $\pi(a), \pi(b) \neq d$.

Note that $a, b \notin R$, since otherwise we would have $a \in h_1^\perp \cap R = \{d\}$ and $b \in h_2^\perp \cap R = \{d\}$, contrary to $a, b \notin H$.

There are no planes on d not in R, therefore $\langle h_1, d\rangle = \langle h_2, d\rangle$. If a were collinear with h_2, then $a^\perp$ would contain $\langle h_1, h_2\rangle$. Since $a \notin R$, this would imply $\pi(a) = d$, against the hypothesis. Therefore a is not collinear with h_2.

Let $S = \mathrm{S}(a, h_2)$. Since $\langle h_1, d\rangle = \langle h_2, d\rangle$, we have $d \in \langle h_1, h_2\rangle \subseteq S$. Let $D = d^\perp \cap S$. Then D is a degenerate hyperplane of S with deep point d. The set of points of S mapped by π to d is contained in D. Since $h_2 \in D - H(S)$ and $h_2 \neq d$, by Lemma 7.14 there exists a point $c \in S - (H(S) \cup D)$ collinear with h_2. By Lemma 7.12 $\Delta|(S - (H(S) \cup D))$ is connected, therefore there exists a walk $W(a, c)$ from a to c inside $S - (H(S) \cup D)$.

The walk C is $\mathcal{C}$-homotopic to the walk $C' = (x_0, x_1, \dots, a) \circ W(a, c) \circ (c, h_2, b, \dots, x_n)$, in which h_2 has no neighbor mapped by π to d. The new part of the walk C, the walk $W(a, c)$, contains no points mapped by π to d.

(T3) Suppose C contains an isolated point h mapped by π to d. That is, $C = (x_0, \dots, a, h, b, \dots, x_n)$, and $\pi(h) = d$ but $\pi(a), \pi(b) \neq d$.

We are going to show that C is $\mathcal{C}$-homotopic to a walk C', obtained from C by replacing the walk (a, h, b) with a walk from a to b containing no points mapped by π to d.

Let $S = \mathrm{S}(a, b)$. We consider three cases.

(T3.1) $a \in b^\perp$.

If $a = b$, then C is homotopic to $C' = (x_0, \dots, a = b, \dots, x_n)$ obtained from C by snapping off the backtrack (a, h, b). If $a \in b^\perp$ and $a \neq b$, then C is homotopic to $C' = (x_0, \dots, a, b, \dots, x_n)$ obtained from C by removing h.

(T3.2) $a \notin b^\perp$ and $S \cap R = \emptyset$.

In this case by Lemma 7.13 the restriction π to S is an isomorphism of S onto R. Therefore, $H'(S) = S \cap \pi^{-1}(H(R))$ is a hyperplane of S. Furthermore, the point h is the unique deep point of $H'(S)$, we have $a, b \in H'(S)$, and a and b are not deep points of $H'(S)$.

Let a' be a point of $S - (H(S) \cup H'(S))$ collinear with a and let $b' \in S - (H(S) \cup H'(S))$ be a point of $S - (H(S) \cup H'(S))$ collinear with b (a' and b' exist by Lemma 7.14). By Lemma 7.12 $\Delta|(S - (H(S) \cup H'(S)))$ is connected, therefore there exists a walk $W(a', b')$ from a' to b' inside it.

The walk C is $\mathcal{C}$-homotopic to the walk $C' = (x_0, \dots, a, a') \circ W(a', b') \circ (b', b, \dots, x_n)$, where the walk $W(a', b')$ contains no points mapped by π to d.

(T3.3) $a \notin b^\perp$ and $S \cap R \neq \emptyset$.

By Lemma 7.5 $S \cap R$ is a line L. In this case the gate in R of every point of S lies on L. Therefore the line L contains d. In particular, $L \subseteq H(R)$. Note that $S \neq R$, since we would have $h = d \in H$ otherwise.

Let $D = d^\perp \cap S$. The set of points of S mapped to d is contained in D. By hypothesis $\pi(a), \pi(b) \neq d$, and $a, b \notin L$ since $L \subseteq H$. Therefore, $a, b \notin D$. By Lemma 7.12 there exists a walk $W(a, b)$ from a to b in $S - (H(S) \cup D)$.

The walk C is $\mathcal{C}$-homotopic to the walk $C' = (x_0, \dots, a) \circ W(a, b) \circ (b, \dots, x_n)$, and $W(a, b)$ contains no points mapped by π to d. This completes the description of (T3).

(T4) Suppose C contains two consecutive points h_1 and h_2 mapped by π into $H(R) - \{d\}$. That is, $C = (x_0, \dots, h_1, h_2, \dots, x_n)$, where $\pi(h_1), \pi(h_2) \in H(R) - \{d\}$.

We are going to show that C is $\mathcal{C}$-homotopic to a walk C', in which (h_1, h_2) has been replaced with a walk from h_1 to h_2 that does not contain two consecutive points mapped by π to $H(R)$. There are two possibilities.

(T4.1) The line $\langle h_1, h_2\rangle$ does not intersect R.

In this case there exists a symplecton S on $\langle h_1, h_2\rangle$ that does not intersect R. This can be seen as follows.

By Lemma 7.5 every symplecton containing $\langle h_1, h_2\rangle$ and intersecting R must contain $\pi(h_1)$ and $\pi(h_2)$. Since the symplecton R contains every plane of Γ on any of its points and $\langle h_1, h_2\rangle \cap R = \emptyset$, we obtain that $\pi(h_1) \neq \pi(h_2)$ and $\mathrm{S}(h_1, h_2, \pi(h_1), \pi(h_2))$ is the unique symplecton of Γ containing $\langle h_1, h_2\rangle$ and intersecting R nontrivially.

The geometry Γ is not equal to R, therefore the line $\langle h_1, h_2\rangle$ is contained in at least two symplecta of Γ. It follows that there is a symplecton S on $\langle h_1, h_2\rangle$ with $S \cap R = \emptyset$.

By Lemma 7.13 the restriction of π to S is an isomorphism of S onto R. Therefore, $H'(S) = S \cap \pi^{-1}(H(R))$ is a hyperplane of S. The deep point of $H'(S)$, if there is one, is the point of S whose gate in R is d. Therefore, by hypothesis h_1 and h_2 are not deep in $H'(S)$. By Lemma 7.14 there exist $c_1, c_2 \in S - (H(S) \cup H'(S))$, such that $c_1 \in h_1^\perp$ and $c_2 \in h_2^\perp$. By Lemma 7.12 there exists a walk $W(c_1, c_2)$ from c_1 to c_2 in $S - (H(S) \cup H'(S))$.

The walk C is $\mathcal{C}$-homotopic to $C' = (x_0, \ldots, h_1, c_1) \circ W(c_1, c_2) \circ (c_2, h_2, \ldots, x_n)$, where $W(c_1, c_2)$ contains no points mapped by π into $H(R)$.

(T4.2) The line $\langle h_1, h_2\rangle$ intersects R at a point u.

In this case $\pi(h_1) = \pi(h_2) = u \in H(R) - \{d\}$.

Let L be a line on u inside R and not in $H(R)$; such a line exists, since by hypothesis $u \neq d$. Every plane of Γ on u is contained in R, therefore $\langle h_1, h_2\rangle$ and L are not contained in a plane. Let $S = \mathrm{S}(\langle h_1, h_2\rangle, L)$. Then $S \cap R = L$ and the gate in R of every point of S lies on L.

Let $U = u^\perp \cap S$. Since $L \cap H(R) = \{u\}$, we have $H'(S) \subseteq U$. Since $h_1, h_2 \neq u$, they are not deep in U. By Lemma 7.14 there exist $c_1, c_2 \in S - (H(S) \cup U)$, such that $c_1 \in h_1^\perp$ and $c_2 \in h_2^\perp$. By Lemma 7.12 there is a walk $W(c_1, c_2)$ from c_1 to c_2 in $S - (H(S) \cup U)$.

The walk C is $\mathcal{C}$-homotopic to the walk $C' = (x_0, \ldots, h_1, c_1) \circ W(c_1, c_2) \circ (c_2, h_2, \ldots, x_n)$, and $W(c_1, c_2)$ contains no points mapped by π to $H(R)$.

(T5) Suppose C contains an isolated point mapped by π to $H(R)$. That is, $C = (x_0, \ldots, a, h, b, \ldots, x_n)$, where $\pi(h) \in H(R)$ and $\pi(a), \pi(b) \notin H(R)$.

We are going to show that C is $\mathcal{C}$-homotopic to a walk C', in which the walk (a, h, b) has been replaced with a walk from a to b containing no points mapped by π to $H(R)$.

We consider three cases.

(T5.1) Suppose $a \in b^\perp$. If $a = b$, then let $C' = (x_0, \ldots, a, \ldots, x_n)$ be the walk obtained by snapping off the backtrack (a, h, b).

If $a \in b^\perp$ and $a \neq b$, then let $C' = (x_0, \ldots, a, b, \ldots, x_n)$.

Suppose now that $a \notin b^\perp$. Let $S = \mathrm{S}(a, b)$. We have $S \neq R$, since $h \in S$ but $h \notin R$ (otherwise we would have $h = \pi(h) \in H$ contrary to $h \notin H$). There are two further possibilities.

(T5.2) S does not intersect R.

In this case by Lemma 7.13 the restriction of π to S is an isomorphism of S onto R and $H'(S) = S \cap \pi^{-1}(H(R))$ is a hyperplane of S. By Lemma 7.12 there exists a walk $W(a, b)$ from a to b inside $S - (H(S) \cup H'(S))$. Then C is $\mathcal{C}$-homotopic to $C' = (x_0, \ldots, a) \circ W(a, b) \circ (b, \ldots, x_n)$, where $W(a, b)$ contains no points mapped by π to $H(R)$.

(T5.3) S intersects R.

Let L be the line $S \cap R$. For every $p \in S - L$, we have $\{\pi(p)\} = p^\perp \cap L$. Therefore, since $\pi(a), \pi(b) \notin H$, the line L is not contained in $H(R)$. Let u denote the unique point of $L \cap H(R)$, and let $U = u^\perp \cap S$. Then U is a hyperplane of S, and $H'(S) \subseteq U$.

The gates of a and b in R are distinct from u, since by hypothesis $\pi(a), \pi(b) \notin H$. Also, a and b do not lie on the line L, since that would force $a, b \in h^\perp \cap R = \{u\} \subseteq H$. Therefore $a, b \notin U$. By Lemma 7.12 there exists a walk $W(a,b)$ from a to b in $S - (H(S) \cup U)$.

The walk C is homotopic to $C' = (x_0, \ldots, a) \circ W(a,b) \circ (b, \ldots, x_n)$, where $W(a,b)$ contains no points mapped by π to $H(R)$. This completes our description of the last transformation (T5).

We now use homotopies (T1)–(T5) to transform C into a walk C_5 that lies in $\Delta - (H \cup H')$. First we repeat (T1) until we obtain a walk C_1 containing no consecutive triples of points mapped by π to d; then we repeat (T2) until a walk C_2 is reached containing no consecutive pairs of points mapped by π to d; then repeating (T3) we obtain a walk C_3 containing no points mapped by π to d at all; then we use (T4), and after that (T5), to obtain a walk C_5, whose image under π lies entirely in $R - H(R)$.

The walk C_5 is a walk in $\Delta - (H \cup H')$ that begins and ends in $R - H$. Therefore by Step 1 the walk C_5 is $\mathcal{C}$-homotopic to a walk lying in $R - H$. This completes the proof of (CH) and the proof of the proposition. □

7.4. Proof of Lemma 2.6

Proof of Lemma 2.6. Statement (i) is clear. We prove (ii). Suppose that $h \in H$ and h is not deep in H.

Let Δ_h be the point-collinearity graph of the geometry $\mathrm{Res}_\Gamma(h) = (\mathcal{L}_h, \Pi_h)$. Let $\phi : h^\perp \to \mathcal{L}_h$ be the map defined by $\phi(x) = \langle h, x\rangle$ for every $x \in h^\perp - H$. Then ϕ is a morphism of $\Delta|h^\perp$ onto Δ_h.

For every line $L \in \mathcal{L}_h$, choose a point $p_L \in L$ and let $X = \{p_L \mid L \in \mathcal{L}_h\}$. Let $\psi : \mathcal{L}_h \to X$ be the map defined by $\psi(L) = p_L$. Then ψ is an isomorphism of Δ_h onto $\Delta|X$. We have $\psi \circ (\phi|X) = \mathrm{id}_{\mathcal{L}_h}$ and $(\phi|X) \circ \psi = \mathrm{id}_X$.

Let H' be the set of all lines of Γ on h contained in H. Then H' is a hyperplane of $\mathrm{Res}_\Gamma(h)$. Let $\mathcal{C}$ be the set of all backtracks and all circular walks of $\Delta_h - H'$ lying inside symplecta of $\mathrm{Res}_\Gamma(h)$. We claim that the image under ψ of a $\mathcal{C}$-homotopy of walks in $\Delta_h - H'$ is a homotopy of walks in $\Delta - H$.

It suffices to prove the statement for an arbitrary $\mathcal{C}$-homotopy of walks (W_1, W_2) in $\Delta_h - H'$, such that $W_1 = A_1 \circ B_1 \circ C_1$, $W_2 = A_2 \circ B_2 \circ C_2$, and $B_1 \circ (B_2^{-1}) \in \mathcal{C}$.

For $i \in \{1,2\}$, let $W_i' = (A_i' \circ B_i' \circ C_i')$ be the image of W_i under ψ, where A_i', B_i', and C_i' are the images of A_i, B_i, and C_i under ψ. Then $B_1' \circ (B_2')^{-1}$ is a circular walk in $\Delta|(X - H)$.

If the walk $B_1 \circ B_2^{-1}$ is a backtrack, then so is $B_1' \circ (B_2')^{-1}$. Suppose that $B_1 \circ B_2^{-1}$ is not a backtrack. Then $B_1 \circ B_2^{-1}$ is contained in a symplecton S of $\mathrm{Res}_\Gamma(h)$.

Let S' be the symplecton of Γ corresponding to S. Then the walk $B_1' \circ (B_2')^{-1}$ is contained in S'. By Theorem 3.1 the graph $\Delta|(S' - H)$ is simply connected, therefore $B_1' \circ (B_2')^{-1}$ is contractible in $\Delta - H$. This proves the claim.

Suppose now that $C = (x_0, \ldots, x_n)$ is a walk in $h^\perp - H$. We need to show that C is contractible in $\Delta - H$.

Let $C' = (\phi(x_0), \ldots, \phi(x_n))$; then C' is a "stammering" circular walk in $\Delta_h - H'$. Let C'' be the circular walk corresponding to C' and let C_0 be the image of C'' under ψ.

The walk C is homotopic to C_0 in $\Delta - H$. The image of C_0 under ϕ is C''. By Propositions 7.2 and 7.3 C'' is $\mathcal{C}$-contractible. Therefore by the claim above C_0 is contractible in $\Delta - H$. It follows that C is contractible in $\Delta - H$. □

8. Existence of Veldkamp lines for hexagonic geometries

In this section we prove Theorem 2.2 by showing that conditions (V1) and (V2) of Section 1.3 hold.

Lemma 4.1 and Proposition 4.2 of Section 4 together show that (V1) holds. The following lemma shows that (V2) holds.

Lemma 8.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a hexagonic geometry, and let Δ denote the point-collinearity graph of Γ. Suppose that Γ has thick lines.*

Let H_1 and H_2 be two distinct hyperplanes of Γ. Then the graph $\Delta|(H_1 - (H_1 \cap H_2))$ is connected.

Remark. Proposition 2.5 shows that the conclusion of Lemma 8.1 holds in the following special case: Γ satisfies condition (A) and at least one of the hyperplanes is of the form $\Delta_2^*(p)$ for some point p of Γ.

Proof of Lemma 8.1. Let x and y be two points in $H_1 - (H_1 \cap H_2)$. We need to show that x and y belong to the same connected component of the graph $\Delta|(H_1 - (H_1 \cap H_2))$. This is clearly true if $x \in y^\perp$. Therefore we can assume $\mathrm{d}_\Delta(x, y) = 2$ or $\mathrm{d}_\Delta(x, y) = 3$.

Case 1. Suppose that $\mathrm{d}_\Delta(x, y) = 2$. Let $U = x^\perp \cap y^\perp$. If $U \cap (H_1 - H_2) \neq \emptyset$ then x and y are in one connected component of the graph $\Delta|(H_1 - (H_1 \cap H_2))$. Therefore, we can assume that $U \cap (H_1 - H_2) = \emptyset$. We consider the following three cases (1.1)–(1.3).

(1.1) Suppose that $U - H_1 \neq \emptyset$.

Let $z \in U - H_1$, and let $Z = z^\perp \cap H_1$. Then Z is a subspace of Γ isomorphic to $\mathrm{Res}_\Gamma(z)$, which is a strong parapolar space with thick lines. Since $x, y \in Z - H_2$, the intersection $Z \cap H_2$ is not all of Z, therefore $Z \cap H_2$ is a hyperplane of Z. By Corollary 3.3 $Z - H_2$ is connected, therefore there is a walk from x to y in $Z - H_2$.

(1.2) Suppose $\{x, y\}$ is a symplectic pair.

Then, since $\mathrm{S}(x, y)$ is a nondegenerate polar space of rank at least 3 with thick lines, the graph $\Delta|(\mathrm{S}(x, y) \cap (H_1 - H_2))$ is connected by the proof of Corollary 5.3 of [15] (or one can apply Corollary 3.3 to the possibly degenerate polar space $\mathrm{S}(x, y) \cap H_1$). Therefore, there is a walk from x to y in $\mathrm{S}(x, y) \cap (H_1 - H_2)$.

(1.3) Suppose that $U \subseteq H_1 \cap H_2$ and $\{x, y\}$ is a special pair.

Let $z \in \mathcal{P}$ be such that $U = \{z\}$. Consider $\mathrm{Res}_\Gamma(z)$. Let Δ_z denote the point-collinearity graph of $\mathrm{Res}_\Gamma(z)$. Let $Z = \mathcal{L}_z$ and, for $i = 1, 2$, let Z_i denote the subspace of $\mathrm{Res}_\Gamma(z)$ consisting of the lines of Γ on z contained in H_i. Then each Z_i is either Z or a hyperplane of $\mathrm{Res}_\Gamma(z)$.

We have $x, y \in z^\perp - H_2$, therefore the space Z_2 is a hyperplane of $\mathrm{Res}_\Gamma(z)$. Let $Z_2' = Z - Z_2$. By Corollary 3.3 the graph $\Delta_z|Z_2'$ is connected.

Suppose first that z is not collinear with any points in $\mathcal{P} - (H_1 \cup H_2)$, that is $z^\perp \subseteq H_1 \cup H_2$. In this case $Z = Z_1 \cup Z_2$. Since $Z_2 \neq Z$, by Theorem 3.2 $Z_1 = Z$. Therefore $Z_2' \subseteq Z_1$. Since $\Delta_z|Z_2'$ is connected, there exists a walk from x to y in $z^\perp - (z^\perp \cap H_2) \subseteq H_1 - (H_1 \cap H_2)$.

Suppose now that z is collinear with at least one point in $\mathcal{P} - (H_1 \cup H_2)$. Let P be a shortest walk from $\langle x, z\rangle$ to $\langle y, z\rangle$ in $\Delta_z|Z_2'$. If the walk P lies in Z_1 we are done.

Suppose P is not contained in Z_1. The edges of the walk P correspond to planes of Γ. Let π_x and π_y be the planes of Γ corresponding to the first and last edges of P. Then π_x contains $\langle x, z\rangle$, and π_y contains $\langle y, z\rangle$.

Suppose $\pi_x \subseteq H_1$ (the case when $\pi_y \subseteq H_1$ is similar). Let L be the line on z shared by π_x and the vertex that follows π_x in the walk P. Let $x' \in L - \{z\}$. If $\{x', y\}$ is a symplectic pair, then x' and y are in one connected component of $\Delta|(H_1 - H_2)$ by case (1.2). If $\{x', y\}$ is a special pair, then $\{x', y\}$ satisfies the hypothesis of case (1.3) with the walk P replaced by a walk of length one less.

The preceding paragraph shows that it suffices to consider the case when $\pi_x, \pi_y \nsubseteq H_1$. Since $\{x, y\}$ is a special pair, by (H2) and (H3) there is exactly one line L in π_y, such that $\langle x, z\rangle$ and L lie in one symplecton of Γ, and are at distance 2 from each other in Δ_z. Since $L \neq \langle y, z\rangle$, we have $L \nsubseteq H_1$.

Let $S = \mathrm{S}(x, L)$, and let $X = x^\perp \cap S \cap H_2$. Then X is a subspace of S isomorphic to $\mathrm{Res}_S(x)$, and $X \cap H_1$ is either all of X or a hyperplane of X. In both cases, since X is a nondegenerate polar space of rank at least 2, the point-collinearity graph of $X \cap H_1$ is not a clique. Therefore, there is a point $z' \in X \cap H_1$ not collinear with z. Then $\langle x, z' \rangle$ is a line in $S \cap H_1$, opposite to L inside S ($z' \notin z^{\perp}$ by the choice of z', and $L \not\subseteq x^{\perp}$ since $d_{\Delta_z}(\langle x, z \rangle, L) = 2$).

Since all lines are thick, there exist a pair of points $v \in \langle x, z' \rangle - H_2$ and $u \in L - H_2$, such that u is collinear with v. Since $v^{\perp} \cap y^{\perp}$ contains $u \in \mathcal{P} - H_1$, by case (1.1) v and y lie in one connected component of $\Delta|(H_1 - (H_1 \cap H_2))$, therefore so do x and y.

Case 2. Suppose $d_{\Delta}(x, y) = 3$. Let $X = \mathrm{Res}_{\Gamma}(x)$, and let Δ_x denote the point-collinearity graph of X. Let H_x be the set of all lines of Γ on x that lie in H_1. Then H_x is a hyperplane of X or all of X. Since X is a strong parapolar space, this implies that the set H_x contains at least two points.

Similarly, we let $Y = \mathrm{Res}_{\Gamma}(y)$, let Δ_y be the point-collinearity graph of Y, and let H_y be the set of all lines of Γ on y that lie in H_1. Then H_y is a hyperplane of Y or all of Y, and H_y contains at least two points.

(2.1) Suppose that $H_x = X$ or $H_y = Y$, or both.

Without loss of generality we can assume that $H_x = X$. Let $L \in H_y$; then $L \subseteq H_1$. Since $d_{\Delta}(x, y) = 3$, by Lemma 4.1 there exists a line M on x opposite to L inside Γ. Since $H_x = X$, we have $M \in H_x$ that is $M \subseteq H_1$. Since all lines are thick, there are points $u \in L - H_2$ and $v \in M - H_2$ such that $d_{\Delta}(u, v) = 2$. By Case 1 u and v lie in one connected component of the graph $\Delta|(H_1 - (H_1 \cap H_2))$, therefore so do x and y.

(2.2) Suppose that H_x and H_y are hyperplanes of X and Y respectively.

Let L be a line in H_x. First, we claim that there is a line M in H_y, such that M is opposite to L, or M is opposite to another line $L' \in H_x$.

By Corollary 3.5 the map $\phi : x^{\perp} \cap \Delta_2^*(y) \to y^{\perp} \cap \Delta_2^*(x)$ that takes each point $u \in x^{\perp} \cap \Delta_2^*(y)$ to the unique point v, such that $\{v\} = u^{\perp} \cap y^{\perp}$, is an isomorphism of point-line spaces induced in Γ on $x^{\perp} \cap \Delta_2^*(y)$ and $y^{\perp} \cap \Delta_2^*(x)$; we denote this isomorphism by the same letter ϕ.

Let $\psi : X \to Y$ be the isomorphism of the point residues of x and y induced by ϕ. That is, for every $u \in x^{\perp} \cap \Delta_2^*(y)$, we have $\psi(\langle x, u \rangle) = \langle y, \phi(u) \rangle$. Let A be a line of Γ on x and let B be a line of Γ on y. If $\psi(A)$ and B are at distance 3 from each other in Y, then A and B are opposite in Γ (cf. the proof of Lemma 4.1).

Let $N = \psi(L)$. Let H_N be the hyperplane of Y consisting of all points at distance at most 2 from N (see axioms (H2) and (H3)).

Suppose $H_y - H_N \neq \emptyset$. Then let $M \in H_y - H_N$. Then $M \subseteq H_1$ and is opposite to L in Γ.

Suppose $H_y \subseteq H_N$. In this case by Corollary 3.3 $H_y = H_N$. In particular $N \in H_y$. As was observed earlier, the set H_x has size at least two. Let L' be any line in H_x distinct from L, and let $N' = \psi(L')$. We claim that H_y contains a point M at distance 3 from N'. That is, there is a line M of Γ on y that lies in H_1 and is opposite to L' in Γ.

If $d_{\Delta_y}(N', N) = 3$, then N is the required line.

Suppose $d_{\Delta_y}(N', N) = 2$. Let (N', A, N) be a geodesic from N' to N in Δ_y. By (H4) it can extended to a geodesic (N', A, N, M) of length 3. Then $M \in H_N = H_y$ and $d_{\Delta_y}(N', M) = 3$.

Suppose $d_{\Delta_y}(N', N) = 1$. Then the lines N' and N span a plane of Γ. By Lemma 7.4 there exists a symplecton S' of Γ containing N' and N. The image of S in Y is a symplecton S of Y. First we extend the walk (N', N) to a geodesic (N', N, A) in S of length 2. Then by (H4) we can extend (N', N, A) to a geodesic (N', N, A, M) of length 3 in Y. We have $M \in H_N = H_y$ and $d_{\Delta_y}(N', M) = 3$. This proves the claim.

Thus, replacing L with another line if necessary, we can assume that there exists a line M in H_y opposite to L. Since all lines have at least 3 points, there are points $u \in L - H_2$ and $v \in M - H_2$, such that $d_{\Delta}(u, v) = 2$. By Case 1 the points u and v belong to one connected component of the graph $\Delta|(H_1 - H_2)$, therefore so do x and y. $\square$

Proof of Theorem 2.2. By Proposition 4.2, combined with Lemma 4.1, and by Lemma 8.1 the hypothesis of Lemma 4.1 of [15] holds (see Section 1.5). Therefore the conclusion follows from Lemma 4.1 of [15]. □

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